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# Kinetic model for the motion of compressible bubbles in a perfect fluid

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## Abstract

Collective behavior of compressible gas bubbles moving in an inviscid incompressible fluid is studied. A kinetic approach is employed, based on an approximate calculation of the fluid flow potential and formulation of Hamilton's equations for generalized coordinates and momenta of bubbles. Kinetic equations governing the evolution of a distribution function of bubbles are derived. These equations are similar to Vlasov's equations. Conservation laws which are direct consequences of the kinetic system are found. It is shown that for a narrowly peaked distribution function they form a closed system of hydrodynamical equations for the mean flow parameters. The system yields the analogue of Rayleigh–Lamb's equation governing oscillations of bubbles. A variational principle for the hydrodynamical system is established and the linear stability analysis is performed. © 2002 Éditions scientifiques et médicales Elsevier SAS. All rights reserved.

**Keywords:** Bubbly fluid; Kinetic equation; Lagrangian

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## 1. Introduction

The motion of a system of compressible bubbles in a fluid is often modeled through averaged equations. The derivation of these equations is the subject of numerous publications. Various models of this type, differing in additional terms in equations, which describe interphase interactions and account for real effects are discussed in monographs by Nakoryakov et al. [19], Nigmatulin [20], Drew and Passman [5] and references therein. Application of the averaging method leads to the difficulties associated with unclosed system of equations. To avoid these difficulties, additional effective equations are used. Most common is to consider the equations for the motion of a single bubble in a fluid as closing constitutive relations (e.g., see Iordansky [12], Kogarko [14], van Wijngaarden [30], Lhuillier [16]).

Other approaches use the methods of statistical physics developed for description of many particle systems. In papers by Biesheuvel and Gorissen [1], Sangani and Didwania [24], Zhang and Prosperetti [33,34], Bulthuis et al. [3], Kok [15] and others, effective hydrodynamical equations and constitutive relations were derived by ensemble averaging.

Kinetic models which give the description of bubbly flows in terms of the one-particle bubble number distribution function have been developed in some recent papers. When the bubbles are considered as rigid massless balls, systems of kinetic equations have been derived by Russo and Smereka [22], Herrero et al. [11] and Jabin and Perthame [13]. Yurkovetsky and Brady [32] have studied the dynamics of bubble clustering. Teshukov [25] has derived an approximate kinetic model for the case of compressible bubbles. Some of the systems of equations obtained are similar to Vlasov's equations used to describe plasma flows. In the derivation of these equations, Hamilton's ordinary differential equations describing the motion of

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a collection of bubbles are employed. The statistical approach can, in principle, give certain basic relations that are postulated on a phenomenological level.

The aim of the present investigation is to derive kinetic and hydrodynamical systems of equations describing the motion of compressible gas bubbles in an inviscid incompressible fluid. We focus on the case of an ideal fluid because such a consideration gives the structure of nonlinear terms for more complicated models.

Outline of the article is as follows. In Section 2 we formulate the system of Hamilton's equations for generalized coordinates of spherical bubbles (spatial coordinates of the centers and radii) and the corresponding impulses. This system is easily written if the potential of the irrotational fluid flow in the region between the bubbles is known. For approximate calculation of the potential, we use an asymptotic expansion of the solution of the Laplace equation in the small parameter – the ratio of the mean radius of the bubbles to the mean distance between them. The Lagrange equations describing the evolution of the bubble system follow from the law of conservation of energy. By a standard transformation, these equations are transformed to Hamilton's equations. In Section 3 we write the Liouville equation for  $N$ -particle distribution function, average this equation over the coordinates and momenta of  $N - 1$  particles, and obtain the analogue of Vlasov's equations for the single-particle distribution function. A hydrodynamical interpretation of equations derived is presented in Section 4. Conservation laws following from the kinetic model are given in Section 5. Here we introduce important average parameters such as the total momentum and the total energy of the gas–liquid system. Hydrodynamical models giving a description of a class of particular solutions of kinetic system, and the analogue of Rayleigh–Lamb's equation are derived in Sections 6, 7. In Section 8 a variational formulation of the hydrodynamical model is presented. The results of linear analysis are discussed in Section 9. In Section 10 the model is modified in order to take into account the external uniform flow of the liquid. Finally, in Section 11 approximate hydrodynamical models are derived and a comparative analysis with classical models is done.

## 2. Lagrangian formulation for a system of $N$ compressible bubbles in an incompressible fluid

We consider  $N$  spherical gas bubbles moving in an unbounded inviscid incompressible fluid. We assume that the fluid flow in the region between bubbles is irrotational and the velocity vector vanishes as  $|\mathbf{x}| \rightarrow \infty$ . The fluid velocity potential  $\varphi(t, \mathbf{x})$  is a solution of the following boundary-value problem:

$$\Delta\varphi = 0, \quad \mathbf{x} \in \Omega = R^3 \setminus \bigcup_j B_j, \quad \left. \frac{\partial\varphi}{\partial n} \right|_{\Gamma_j} = \mathbf{v}_j \cdot \mathbf{n}_j + s_j, \quad \nabla\varphi \rightarrow 0, \quad |\mathbf{x}| \rightarrow \infty. \quad (2.1)$$

Here  $\mathbf{v}_j(t) = \mathbf{x}'_j(t)$  are the velocities of the centers of spherical bubbles,  $s_j = b'_j(t)$  are the velocities of expansion of the bubbles,  $\mathbf{x}_j(t)$  are the radius-vectors of the centers,  $b_j(t)$  are the radii of the bubbles ( $j = 1, \dots, N$ ),  $B_j$  and  $\Gamma_j$  are a ball and a sphere of radius  $b_j(t)$  with center at the point  $\mathbf{x} = \mathbf{x}_j(t)$ , respectively, and  $\mathbf{n}_j$  is the unit normal vector to  $\Gamma_j$  directed to the fluid. Prime denotes derivative with respect to time. This problem describes the motion of a collection of compressible bubbles in otherwise quiescent fluid.

The unknown potential of the irrotational flow can be written as

$$\varphi = \sum_{j=1}^N (\mathbf{v}_j \cdot \boldsymbol{\psi}_j + s_j \varphi_j), \quad (2.2)$$

where, in view of (2.1), the harmonic functions  $\boldsymbol{\psi}_j(t, \mathbf{x})$  and  $\varphi_j(t, \mathbf{x})$  satisfy the conditions

$$\left. \frac{\partial \boldsymbol{\psi}_j}{\partial n} \right|_{\Gamma_k} = \delta_{jk} \mathbf{n}_j, \quad \left. \frac{\partial \varphi_j}{\partial n} \right|_{\Gamma_k} = \delta_{jk} \quad (2.3)$$

and their gradients vanish at infinity. Here  $\delta_{jk}$  are the Kronecker symbols.

Let  $\bar{b}$  be the mean bubble radius and  $d$  the mean distance between bubbles. In what follows, we consider a rarefied bubbly fluid for which  $\beta = \bar{b}/d$  is a small parameter. Using asymptotic expansions in the small parameter, we obtain approximate values of the flow potential near each bubble.

The function  $\varphi_{j0} = -b_j^2/r_j$  ( $r_j = |\mathbf{x} - \mathbf{x}_j|$ ), describing the flow induced by monopole, is harmonic outside the ball  $B_j$ , satisfies the condition  $\partial\varphi_{j0}/\partial n = 1$  on  $\Gamma_j$  and the decay condition for large  $|\mathbf{x}|$ . In the neighborhood of the  $i$ th bubble, we have

$$r_j^2 = r_{ij}^2 |\mathbf{n}_{ji} - r_{ij}^{-1}(\mathbf{x} - \mathbf{x}_i)|^2 = r_{ij}^2 (1 - 2 \cos \theta_{ji} (r_i/r_{ij}) + (r_i/r_{ij})^2),$$

$$r_{ij} = |\mathbf{x}_i - \mathbf{x}_j|, \quad \mathbf{n}_{ji} = (\mathbf{x}_j - \mathbf{x}_i) r_{ij}^{-1}, \quad \cos \theta_{ji} = (\mathbf{x} - \mathbf{x}_i) \cdot \mathbf{n}_{ji} r_i^{-1} \quad \text{and} \quad r_i = |\mathbf{x} - \mathbf{x}_i|.$$

Using the generating function

$$\frac{1}{\sqrt{1-2\zeta\cos\theta+\zeta^2}} = \sum_{n=0}^{\infty} P_n(\cos\theta)\zeta^n$$

for Legendre polynomials  $P_n(\cos\theta)$  [10] and the representation for  $r_j$  written above, we obtain the following series expansion of the function  $\varphi_{j0}$ :

$$\varphi_{j0} = -\frac{b_j^2}{r_{ij}} \sum_{n=0}^{\infty} \left(\frac{r_i}{b_i}\right)^n \left(\frac{b_i}{r_{ij}}\right)^n P_n(\cos\theta_{ji}), \quad (2.4)$$

which is valid in the neighborhood of the  $i$ th bubble ( $r_i/r_{ij} < 1$ ). Eq. (2.4) for  $r_i = b_i$  gives the expansion of the trace of the function  $\varphi_{j0}$  on  $\Gamma_i$  in powers of  $b_i/r_{ij}$ . We note that the quantity  $b_i/r_{ij}$  is of order  $\beta \ll 1$ . It is known that, along with  $r^n P_n(\cos\theta)$ , the function  $r^{-(n+1)} P_n(\cos\theta)$  also satisfies the Laplace equation [10]. Consider the harmonic function

$$\varphi_{ji} = -\frac{b_j^2}{r_{ij}} \sum_{n=1}^{\infty} \frac{n}{n+1} \left(\frac{b_i}{r_i}\right)^{n+1} \left(\frac{b_i}{r_{ij}}\right)^n P_n(\cos\theta_{ji}). \quad (2.5)$$

Coefficients of this series are chosen in such a way that the condition

$$\frac{\partial}{\partial r_i}(\varphi_{j0} + \varphi_{ji}) = 0 \quad (2.6)$$

is satisfied on  $\Gamma_i$ . Let us introduce the function

$$\varphi_j = \varphi_{j0} + \sum_{i \neq j} \varphi_{ji}^{(2)}. \quad (2.7)$$

Here  $\varphi_{ji}^{(2)}$  is defined by formula (2.5) where the total sum of the series is replaced by the partial sum of terms up to  $n = 2$ . It is easy to see that  $\partial\varphi_j/\partial n|_{\Gamma_k} = \delta_{jk} + O(\beta^4)$  by virtue of Eqs. (2.6), (2.7) and the estimates  $\varphi_{ji}|_{\Gamma_k} = O(\beta^4)$  and  $\nabla\varphi_{ji}|_{\Gamma_k} = O(\beta^4)$ ,  $i \neq k$ . Hence, the function  $\varphi_j$  defined in (2.7) is an approximate solution of problem (2.3) for the Laplace equation which is correct to  $\beta^3$ .

To find  $\psi_j$  we consider the harmonic function describing the flow induced by dipole

$$\psi_{j0} = \frac{b_j^3}{2} \nabla(r_j^{-1}) = -\frac{1}{2} \left(\frac{b_j}{r_j}\right)^3 (\mathbf{x} - \mathbf{x}_j)$$

which satisfies the boundary condition

$$\frac{\partial\psi_{j0}}{\partial n} \Big|_{\Gamma_j} = \mathbf{n}_j$$

on the surface of the  $j$ th bubble. In the neighborhood of the  $i$ th bubble,  $\psi_{j0}$  admits the following approximate representation by a partial Taylor series with the remainder of order  $\beta^4$ :

$$\psi_{j0} = \frac{b_j}{2} \left( \left(\frac{b_j}{r_{ij}}\right)^2 \mathbf{n}_{ji} - \left(\frac{b_j}{r_{ij}}\right)^2 \left(\frac{r_i}{b_i}\right) \left(\frac{b_i}{r_{ij}}\right) B_{ij} \left(\frac{\mathbf{x} - \mathbf{x}_i}{r_i}\right) \right) + O(\beta^4). \quad (2.8)$$

where  $B_{ij} = I - 3\mathbf{n}_{ji} \otimes \mathbf{n}_{ji}$ ,  $I$  is the unit matrix, and  $\mathbf{a} \otimes \mathbf{b}$  is the dyad tensor. The harmonic function (which is constructed in the same way as  $\varphi_{ji}$ )

$$\psi_{ji} = \frac{b_j}{4} \left(\frac{b_j}{r_{ij}}\right)^2 \left(\frac{b_i}{r_{ij}}\right) \left(\frac{b_i}{r_i}\right)^2 B_{ij} \frac{\mathbf{x} - \mathbf{x}_i}{r_i} \quad (2.9)$$

satisfies the conditions  $\partial(\psi_{j0} + \psi_{ji})/\partial n|_{\Gamma_i} = 0$ ,  $\psi_{ji}|_{\Gamma_k} = O(\beta^4)$  and  $\nabla\psi_{ji}|_{\Gamma_k} = O(\beta^4)$  for  $i \neq k$ . Using these equalities we conclude that

$$\psi_j = \psi_{j0} + \sum_{i \neq j} \psi_{ji} \quad (2.10)$$

is an approximate solution of problem (2.3) which is correct to  $\beta^3$  (see also Russo and Smereka [22]). This order of approximation exactly accounts for pair interactions of the bubbles. In the next terms of expansion we have to consider

contributions of  $\psi_{ji}$  and  $\varphi_{ij}$  on  $\Gamma_k$  that corresponds to accounting for triple interactions, etc. Notice that if  $N$  is the number of the bubbles in a volume  $L^3$  then  $N^{1/3} \approx L/d$  and void fraction  $\alpha \approx N\bar{b}^3/L^3 \approx \bar{b}^3/d^3 = \beta^3$ . We conclude that our approximation is correct to order  $O(\alpha)$ .

As a result, we have obtained an approximate representation of the fluid velocity potential for the specified positions  $\mathbf{x}_j$ , radii  $b_j$ , translational velocities  $\mathbf{v}_j$  and dilatation velocities  $s_j$  of bubbles. This enables us to calculate the kinetic energy of the fluid:

$$T = \frac{\rho}{2} \iint_{\Omega} |\mathbf{u}|^2 d\Omega = -\frac{\rho}{2} \sum_{i=1}^N \iint_{\Gamma_i} \varphi \frac{\partial \varphi}{\partial n} d\Gamma = -\frac{\rho}{2} \sum_{i=1}^N \iint_{\Gamma_i} \varphi (\mathbf{v}_i \cdot \mathbf{n}_i + s_i) d\Gamma.$$

Here  $\rho = \text{const}$  is the density of the fluid, the normal vector  $\mathbf{n}_i$  is directed to the fluid. Using (2.2), we can rewrite  $T$  as

$$T = -\frac{\rho}{2} \sum_{j=1}^N \sum_{i=1}^N ((\mathbf{v}_j \cdot A_{ji} \mathbf{v}_i) + s_j \mathbf{d}_{ij} \cdot \mathbf{v}_i + \mathbf{v}_j \cdot \mathbf{c}_{ij} s_i + s_j e_{ij} s_i),$$

$$A_{ji} = \iint_{\Gamma_i} \psi_j \otimes \mathbf{n}_i d\Gamma, \quad \mathbf{d}_{ij} = \iint_{\Gamma_i} \varphi_j \mathbf{n}_i d\Gamma, \quad \mathbf{c}_{ij} = \iint_{\Gamma_i} \psi_j d\Gamma, \quad e_{ij} = \iint_{\Gamma_i} \varphi_j d\Gamma. \quad (2.11)$$

In evaluation of the coefficients of this quadratic form we use approximate expressions (2.7), (2.10) for the potentials  $\psi_j$  and  $\varphi_j$  on  $\Gamma_i$ . The coefficients of quadratic form (2.11) are calculated in Appendix A. This gives the following expression for the kinetic energy of the fluid:

$$T = \frac{\rho}{2} \left( \sum_{i=1}^N b_i^3 \left( \frac{2}{3} \pi |\mathbf{v}_i|^2 + 4\pi s_i^2 \right) + \sum_{i=1}^N \sum_{j \neq i}^N \left( 4\pi b_i^2 b_j \frac{b_j}{r_{ij}} s_i s_j + 2\pi b_i^3 \left( \frac{b_j}{r_{ij}} \right)^2 s_j (\mathbf{n}_{ji} \cdot \mathbf{v}_i) + 2\pi b_j^3 \left( \frac{b_i}{r_{ij}} \right)^2 s_i (\mathbf{n}_{ij} \cdot \mathbf{v}_j) + \pi b_i^3 \left( \frac{b_j}{r_{ij}} \right)^3 (\mathbf{v}_j \cdot B_{ij} \mathbf{v}_i) \right) \right). \quad (2.12)$$

In this formula, the first sum comprises the zero-order terms in  $\beta$  and the second sum includes terms of orders  $\beta$ ,  $\beta^2$ , and  $\beta^3$ . An analogous formula for kinetic energy containing the terms  $O(\beta^2)$  has been also obtained by Voinov and Golovin [27] and Garipov [7]. For the case of rigid bubbles of equal radii considered in the paper by Russo and Smereka [22], the formula for kinetic energy contained only the zeroth and the third-order terms.

In the calculation of the total kinetic energy of the system “fluid–gas bubbles”, we do not take account of the kinetic energy of the gas, because the gas density is considerably smaller than the fluid density. The law of conservation of energy for the fluid that occupies the region between the bubbles is written as

$$\frac{dT}{dt} = \sum_{i=1}^N \iint_{\Gamma_i} (p - P_{\infty}) u_n d\Gamma.$$

Here  $p$  is the pressure in the fluid,  $P_{\infty} = \text{const}$  is the pressure at infinity, and  $u_n = \mathbf{u} \cdot \mathbf{n} = \partial \varphi / \partial n$ .

We note that the assumption of sphericity of the bubbles used for an approximate description of the flow simplifies substantially the problem and makes it possible to determine the main contribution to the variation of the fluid flow potential caused by oscillations of the bubble volume. In the exact formulation of the problem, the bubble shape must be obtained as a result of solving the problem with unknown fluid–gas boundary from the condition of equality of the pressures in the gas and in the fluid on  $\Gamma_i$ . Therefore, in the approximate description, the formulation of the problem is modified: on the boundary we require equality of the pressures averaged over the surface.

Let  $\tau$  be the volume of a bubble. The state of the gas inside of a bubble will be described approximately under the assumption that the gas density and the gas pressure are constant over the volume. We assume that, initially, all bubbles have the same mass and temperature. Then they have equal masses at any time and, hence, the pressure  $p_g$  in a bubble is determined only by its volume  $\tau$ :  $p_g = p_g(\tau)$ . For the bubbles of equal mass the internal pressure may be a two parameter function (in the case, for example, when the initial temperature in bubbles differs). To simplify the analysis which follows, we consider here the one parameter case. Using (2.1), we obtain

$$\sum_{i=1}^N \iint_{\Gamma_i} (p - P_{\infty}) \frac{\partial \varphi}{\partial n} d\Gamma = \sum_{i=1}^N \left( \mathbf{v}_i \iint_{\Gamma_i} (p - P_{\infty}) \mathbf{n} d\Gamma + s_i \iint_{\Gamma_i} (p - P_{\infty}) d\Gamma \right). \quad (2.13)$$

The integral in the first term of the right-hand side of this equality is zero (the total force exerted on a bubble is equal to zero by virtue of the law of conservation of momentum). Then taking into account that the pressure depends only on the bubble volume, we can write Eq. (2.13) in the form

$$\sum_{i=1}^N \iint_{\Gamma_i} (p - P_\infty) \frac{\partial \varphi}{\partial n} d\Gamma = \sum_{i=1}^N (p_g(\tau_i) - P_\infty) \frac{d\tau_i}{dt} = - \sum_{i=1}^N \frac{d\varepsilon(\tau_i)}{dt}, \quad \varepsilon(\tau) = \int_{\tau}^{\infty} p_g(\tau) d\tau + P_\infty \tau.$$

It is well known that the problem of motion of a fluid with bubbles is Lagrangian in the case where the flow of the fluid is completely determined by bubble motion [27]. Exactly this situation is considered in the present paper. Using the law of conservation of energy, we obtain the Euler–Lagrange equations for the generalized coordinates and corresponding velocities:

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\mathbf{y}}_i} \right) - \frac{\partial L}{\partial \mathbf{y}_i} = 0, \quad L = T - U, \quad U = \sum_{i=1}^N \varepsilon(\tau_i). \quad (2.14)$$

Here  $\mathbf{y}_i = (\mathbf{x}_i, b_i)^T$  ( $i = 1, \dots, N$ ) are the vectors with four components.

To convert the Euler–Lagrange equations to the Hamiltonian form, we introduce the Hamiltonian

$$H = T + U \quad (2.15)$$

and generalized impulses  $\mathbf{p}_i$  and  $q_i$  related to generalized coordinates  $\mathbf{x}_i$  and  $b_i$ :

$$\mathbf{p}_i = \frac{\partial T}{\partial \mathbf{v}_i} = -\rho \int_{\Gamma_i} \varphi \mathbf{n} d\Gamma, \quad q_i = \frac{\partial T}{\partial s_i} = -\rho \int_{\Gamma_i} \varphi d\Gamma. \quad (2.16)$$

Notice that

$$T = -\frac{\rho}{2} \sum_{i=1}^N \iint_{\Gamma_i} \varphi (\mathbf{v}_i \cdot \mathbf{n}_i + s_i) d\Gamma = \frac{1}{2} \sum_{i=1}^N (\mathbf{v}_i \cdot \mathbf{p}_i + s_i q_i)$$

where  $\mathbf{p}_i, q_i$  are given in (2.16). Differentiating approximate expression for kinetic energy (2.12) we obtain explicit formulae for  $\mathbf{p}_i$  and  $q_i$ :

$$\begin{aligned} \mathbf{p}_i &= \rho \left( \frac{2}{3} \pi b_i^3 \mathbf{v}_i + 2\pi b_i^3 \sum_{j \neq i} \left( \frac{b_j}{r_{ij}} \right)^2 s_j \mathbf{n}_{ji} + \pi b_i^3 \sum_{j \neq i} \left( \frac{b_j}{r_{ij}} \right)^3 B_{ij} \mathbf{v}_j \right), \\ q_i &= \rho \left( 4\pi b_i^3 s_i + 4\pi b_i^2 \sum_{j \neq i} \frac{b_j^2 s_j}{r_{ij}} - 2\pi b_i^2 \sum_{j \neq i} \frac{b_j^3 (\mathbf{n}_{ji} \cdot \mathbf{v}_j)}{r_{ij}^2} \right). \end{aligned} \quad (2.17)$$

In Hamiltonian form, the equations of the bubble motion are:

$$\frac{d\mathbf{x}_i}{dt} = H_{\mathbf{p}_i}, \quad \frac{d\mathbf{p}_i}{dt} = -H_{\mathbf{x}_i}, \quad \frac{db_i}{dt} = H_{q_i}, \quad \frac{dq_i}{dt} = -H_{b_i}, \quad (2.18)$$

where  $H$  is given in (2.15).

### 3. Derivation of kinetic equations

In the present paper, we consider the collisionless case assuming that the bubbly medium is sufficiently rarefied, so that over the characteristic time the bubbles collide with each other very rarely. The basic equation of the kinetic model is the law of conservation of the number of particles during their motion. Consider the  $N$ -particle distribution function

$$f^{(N)}(t, \mathbf{x}_1, b_1, \mathbf{p}_1, q_1, \dots, \mathbf{x}_N, b_N, \mathbf{p}_N, q_N)$$

which, in view of (2.18), satisfies the Liouville equation [17]

$$f_t^{(N)} + \sum_{k=1}^N \left( \operatorname{div}_{\mathbf{x}_k} (H_{\mathbf{p}_k} f^{(N)}) - \operatorname{div}_{\mathbf{p}_k} (H_{\mathbf{x}_k} f^{(N)}) + (H_{q_k} f^{(N)})_{b_k} - (H_{b_k} f^{(N)})_{q_k} \right) = 0. \quad (3.1)$$

We assume that the distribution function vanishes for large  $\mathbf{x}_k$ ,  $b_k$ ,  $\mathbf{p}_k$ , and  $q_k$ . Eq. (3.1) is an analogue of the hydrodynamic continuity equation for the motion along the trajectories of system (2.18). The unknown function in this equation depends on a large number of independent variables, and construction of its solution is, in fact, equivalent to integration of Eqs. (2.18). Therefore, using the averaging method, we deduce simpler equations that describe the evolution of the single-particle distribution function defined by

$$f_j^{(1)}(t, \mathbf{x}_j, b_j, \mathbf{p}_j, q_j) = \int f^{(N)} d\Omega_{N-1,j}. \quad (3.2)$$

Hereafter we use the notations

$$d\Omega_i = d\mathbf{p}_i dq_i db_i, \quad d\Omega_{N-1,j} = \prod_{i \neq j} d\Omega_i d\mathbf{x}_i, \quad d\Omega_{N-2,j,k} = \prod_{i \neq j,k} d\Omega_i d\mathbf{x}_i, \\ f_{j,k}^{(2)}(t, \mathbf{x}_j, b_j, \mathbf{p}_j, q_j, \mathbf{x}_k, b_k, \mathbf{p}_k, q_k) = \int f^{(N)} d\Omega_{N-2,j,k}.$$

In the following we assume that for any  $j, k$

$$f_j^{(1)}(t, \mathbf{x}, b, \mathbf{p}, q) = f_k^{(1)}(t, \mathbf{x}, b, \mathbf{p}, q) = f^{(1)}(t, \mathbf{x}, b, \mathbf{p}, q), \\ f_{j,k}^{(2)}(t, \mathbf{x}_j, b_j, \mathbf{p}_j, q_j, \mathbf{x}_k, b_k, \mathbf{p}_k, q_k) = f_j^{(1)}(t, \mathbf{x}_j, b_j, \mathbf{p}_j, q_j) f_k^{(1)}(t, \mathbf{x}_k, b_k, \mathbf{p}_k, q_k). \quad (3.3)$$

Assumptions (3.3) mean that the distribution of different bubbles is described by the same single-particle distribution function and that the hypothesis of “molecular chaos” is valid (see also [22]). Multiplying Eq. (3.1) by  $d\Omega_{N-1,1}$  and integrating over the generalized coordinates and the momenta of  $N-1$  particles, we obtain the following equation for the function  $f_1^{(1)}$ :

$$f_{1t}^{(1)} + \operatorname{div}_{\mathbf{x}_1} \left( \int H_{\mathbf{p}_1} f^{(N)} d\Omega_{N-1,1} \right) - \operatorname{div}_{\mathbf{p}_1} \left( \int H_{\mathbf{x}_1} f^{(N)} d\Omega_{N-1,1} \right) + \left( \int H_{q_1} f^{(N)} d\Omega_{N-1,1} \right)_{b_1} \\ - \left( \int H_{b_1} f^{(N)} d\Omega_{N-1,1} \right)_{q_1} = 0. \quad (3.4)$$

All other divergence terms are integrated to zero because  $f^{(N)}$  vanishes for great values of arguments. Let us calculate the integral terms of this equation. Since  $H_{\mathbf{p}_1} = \mathbf{v}_1$ ,  $H_{q_1} = s_1$ , we have to express integrals

$$\int \mathbf{v}_1 f^{(N)} d\Omega_{N-1,1} \quad \text{and} \quad \int s_1 f^{(N)} d\Omega_{N-1,1}$$

in terms of  $\mathbf{x}_1$ ,  $b_1$ ,  $\mathbf{p}_1$ ,  $q_1$ .

If we multiply Eqs. (2.17) by  $f^{(N)} d\Omega_{N-1,i}$  and integrate with respect to variables  $\mathbf{x}_j$ ,  $b_j$ ,  $\mathbf{p}_j$ ,  $q_j$  having indices  $j \neq i$ , then we obtain the relations

$$\int \mathbf{v}_i f^{(N)} d\Omega_{N-1,i} = \frac{2\mathbf{p}_i}{\rho\tau_i} f_i^{(1)} - 3 \sum_{j \neq i} \int \nabla_{\mathbf{x}_i} \left( \frac{1}{r_{ij}} \right) \left( \int b_j^2 \left( \int s_j f^{(N)} d\Omega_{N-2,i,j} \right) d\Omega_j \right) d\mathbf{x}_j \\ + \frac{3}{2} \lim_{\varepsilon \rightarrow 0} \sum_{j \neq i} \int_{|\mathbf{x}_i - \mathbf{x}_j| > \varepsilon} \frac{\partial^2}{\partial \mathbf{x}_i^2} \left( \frac{1}{r_{ij}} \right) \left( \int b_j^3 \left( \int \mathbf{v}_j f^{(N)} d\Omega_{N-2,i,j} \right) d\Omega_j \right) d\mathbf{x}_j, \\ \int s_i f^{(N)} d\Omega_{N-1,i} = \frac{q_i}{3\rho\tau_i} f_i^{(1)} - \frac{1}{b_i} \sum_{j \neq i} \int \frac{1}{r_{ij}} \left( \int b_j^2 \left( \int s_j f^{(N)} d\Omega_{N-2,i,j} \right) d\Omega_j \right) d\mathbf{x}_j \\ + \frac{1}{2b_i} \sum_{j \neq i} \int \nabla_{\mathbf{x}_i} \left( \frac{1}{r_{ij}} \right) \left( \int b_j^3 \left( \int \mathbf{v}_j f^{(N)} d\Omega_{N-2,i,j} \right) d\Omega_j \right) d\mathbf{x}_j. \quad (3.5)$$

Here we have to find the principal values of improper integrals.

Let us show that for any given distribution function Eqs. (3.5) determine functions  $\tilde{\mathbf{v}}(t, \mathbf{x}, b, \mathbf{p}, q)$  and  $\tilde{s}(t, \mathbf{x}, b, \mathbf{p}, q)$  satisfying the equalities

$$\int \mathbf{v}_j f^{(N)} d\Omega_{N-2,i,j} = \tilde{\mathbf{v}}(t, \mathbf{x}_j, b_j, \mathbf{p}_j) f_{i,j}^{(2)}, \quad \int s_j f^{(N)} d\Omega_{N-2,i,j} = \tilde{s}(t, \mathbf{x}_j, b_j, \mathbf{p}_j, q_j) f_{i,j}^{(2)}.$$

They represent the mean translational and dilatational bubble velocities. After substituting of these expressions into Eqs. (3.5) by virtue of properties (3.3) we get

$$\begin{aligned}\tilde{\mathbf{v}}_i f_i^{(1)} &= \left( \frac{2\mathbf{p}_i}{\rho\tau_i} - 3 \sum_{j \neq i} \int \nabla_{\mathbf{x}_i} \left( \frac{1}{r_{ij}} \right) \left( \int b_j^2 \tilde{s}_j f_j^{(1)} d\Omega_j \right) d\mathbf{x}_j \right. \\ &\quad \left. + \frac{3}{2} \lim_{\varepsilon \rightarrow 0} \sum_{j \neq i} \int_{|\mathbf{x}_i - \mathbf{x}_j| > \varepsilon} \frac{\partial^2}{\partial \mathbf{x}_i^2} \left( \frac{1}{r_{ij}} \right) \left( \int b_j^3 \tilde{\mathbf{v}}_j f_j^{(1)} d\Omega_j \right) d\mathbf{x}_j \right) f_i^{(1)}, \\ \tilde{s}_i f_i^{(1)} &= \left( \frac{q_i}{3\rho\tau_i} - \frac{1}{b_i} \sum_{j \neq i} \int \frac{1}{r_{ij}} \left( \int b_j^2 \tilde{s}_j f_j^{(1)} d\Omega_j \right) d\mathbf{x}_j + \frac{1}{2b_i} \sum_{j \neq i} \int \nabla_{\mathbf{x}_i} \left( \frac{1}{r_{ij}} \right) \left( \int b_j^3 \tilde{\mathbf{v}}_j f_j^{(1)} d\Omega_j \right) d\mathbf{x}_j \right) f_i^{(1)},\end{aligned}$$

where  $\tilde{\mathbf{v}}_j = \tilde{\mathbf{v}}(t, \mathbf{x}_j, b_j, \mathbf{p}_j, q_j)$  and  $\tilde{s}_j = \tilde{s}(t, \mathbf{x}_j, b_j, \mathbf{p}_j, q_j)$ .

Let us introduce  $f = N f^{(1)}$  which is the density function in phase space satisfying the equalities

$$\int f d\Omega = n, \quad \int f d\Omega d\mathbf{x} = N,$$

where  $n$  is the number of bubbles per unit volume,  $d\Omega = d\mathbf{p} dq db$ . We define the new functions

$$\begin{aligned}\varphi(t, \mathbf{x}) &= -\frac{1}{4\pi} \int \frac{1}{|\mathbf{x} - \mathbf{x}_j|} \left( 4\pi \int b_j^2 \tilde{s}_j f_j d\Omega_j \right) d\mathbf{x}_j, \\ \boldsymbol{\psi}(t, \mathbf{x}) &= -\frac{1}{4\pi} \int \frac{1}{|\mathbf{x} - \mathbf{x}_j|} \left( -2\pi \int b_j^3 \tilde{\mathbf{v}}_j f_j d\Omega_j \right) d\mathbf{x}_j,\end{aligned}$$

where  $j$  is any index. We use the designation  $f_j$  for  $f(\mathbf{x}_j, b_j, \mathbf{p}_j, q_j)$ . Note that  $-(4\pi|\mathbf{x} - \mathbf{x}_j|)^{-1}$  is the Green function for Laplace equation in  $R^3$ . By using the indistinguishability of particles we write

$$\begin{aligned}\tilde{\mathbf{v}}(t, \mathbf{x}_i, b_i, \mathbf{p}_i, q_i) &= \frac{2\mathbf{p}_i}{\rho\tau_i} + 3 \left( \nabla(\varphi(t, \mathbf{x}_i) + \operatorname{div} \boldsymbol{\psi}(t, \mathbf{x}_i)) + \frac{3}{2} \int \tau \tilde{\mathbf{v}}(t, \mathbf{x}_i, b, \mathbf{p}, q) f(t, \mathbf{x}_i, b, \mathbf{p}, q) d\Omega \right) \frac{N-1}{N}, \\ \tilde{s}(t, \mathbf{x}_i, b_i, \mathbf{p}_i, q_i) &= \frac{q_i}{3\rho\tau_i} + \frac{1}{b_i} (\varphi(t, \mathbf{x}_i) + \operatorname{div} \boldsymbol{\psi}(t, \mathbf{x}_i)) \frac{N-1}{N}.\end{aligned}\quad (3.6)$$

In evaluation of the limit in (3.6) we have applied the formula (see also Russo and Smereka [22]):

$$\lim_{\varepsilon \rightarrow 0} \left( -\frac{1}{4\pi} \int_{|\mathbf{x} - \mathbf{y}| > \varepsilon} \nabla_{\mathbf{x}} \operatorname{div}_{\mathbf{x}} \left( \frac{\mathbf{a}(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|} \right) d\mathbf{y} \right) = -\frac{1}{4\pi} \nabla \operatorname{div} \left( \int \frac{\mathbf{a}(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|} d\mathbf{y} \right) - \frac{1}{3} \mathbf{a}(\mathbf{x}). \quad (3.7)$$

Passing to the limit  $N \rightarrow \infty$  we obtain a system of equations determining unknown functions  $\tilde{\mathbf{v}}(t, \mathbf{x}, b, \mathbf{p}, q)$  and  $\tilde{s}(t, \mathbf{x}, b, \mathbf{p}, q)$

$$\begin{aligned}\tilde{\mathbf{v}} &= \frac{2\mathbf{p}}{\rho\tau} + 3 \left( \nabla\Phi + \frac{1}{2} \int \tau \tilde{\mathbf{v}} f d\Omega \right), \quad \tilde{s} = \frac{q}{3\rho\tau} + \frac{\Phi}{b}, \\ \Phi &= \varphi + \phi, \quad \Delta\varphi = 4\pi \int b^2 \tilde{s} f d\Omega, \quad \Delta\phi = -\frac{3}{2} \operatorname{div} \left( \int \tau \tilde{\mathbf{v}} f d\Omega \right),\end{aligned}\quad (3.8)$$

where  $\phi = \operatorname{div} \boldsymbol{\psi}$ ,  $d\Omega = d\mathbf{p} dq db$  and  $\tau = 4/3\pi b^3$  is the volume of the bubble. To obtain the expressions for functions  $\tilde{\mathbf{v}}$  and  $\tilde{s}$  in terms of  $\mathbf{x}$ ,  $b$ ,  $\mathbf{p}$ ,  $q$ , we shall convert Eqs. (3.8). If we multiply the first equality in (3.8) by  $\tau f$ , the second by  $b^2 f$  and integrate over  $d\Omega$ , we obtain linear equations which allow us to find

$$\begin{aligned}-\frac{3}{2} \int \tau \tilde{\mathbf{v}} f d\Omega &= \frac{1}{3\alpha - 2} \left( \frac{6}{\rho} \int \mathbf{p} f d\Omega + 9\alpha \nabla\Phi \right), \\ 4\pi \int b^2 \tilde{s} f d\Omega &= \frac{1}{\rho} \int \frac{q}{b} f d\Omega + 4\pi \Phi \int d\Omega,\end{aligned}\quad (3.9)$$

where  $\alpha = \int \tau f d\Omega$  is the void fraction of bubbles. Using Eqs. (3.8) and (3.9) we arrive to formulae defining functions  $\tilde{\mathbf{v}}$ ,  $\tilde{s}$  depending on  $\mathbf{p}$ ,  $q$ ,  $b$ ,  $\mathbf{x}$ :

$$\tilde{\mathbf{v}} = \frac{2\mathbf{p}}{\rho\tau} - \frac{6}{3\alpha - 2} \left( \nabla\Phi + \frac{1}{\rho} \int \mathbf{p} f d\Omega \right), \quad \tilde{s} = \frac{q}{3\rho\tau} + \frac{\Phi}{b}, \quad (3.10)$$

where  $\Phi$  satisfies the equation

$$\Delta\Phi = \operatorname{div} \left( \frac{6}{\rho(3\alpha - 2)} \int \mathbf{p} f d\Omega + \frac{9\alpha}{3\alpha - 2} \nabla\Phi \right) + 4\pi \Phi \int b f d\Omega + \frac{1}{\rho} \int \frac{q}{b} f d\Omega.$$

Using formulas obtained we find the integrals in the second and the fourth terms of Eq. (3.4):

$$\int H_{\mathbf{p}_1} f^{(N)} d\Omega_{N-1,1} = \int \mathbf{v}_1 f^{(N)} d\Omega_{N-1,1} = \left( \frac{2\mathbf{p}_1}{\rho\tau_1} - \frac{6}{3\alpha_1 - 2} \left( \nabla\Phi + \frac{1}{\rho} \int \mathbf{p} f d\Omega \right) \right) f_1^{(1)},$$

$$\int H_{q_1} f^{(N)} d\Omega_{N-1,1} = \int s_1 f^{(N)} d\Omega_{N-1,1} = \left( \frac{q_1}{3\rho\tau_1} + \frac{\Phi}{b_1} \right) f_1^{(1)}.$$

To derive the kinetic equation for the density function in phase space  $f(t, \mathbf{x}, b, \mathbf{p}, q)$ , we have also to find the integrals

$$\int H_{\mathbf{x}_1} f^{(N)} d\Omega_{N-1,1}, \quad \int H_{b_1} f^{(N)} d\Omega_{N-1,1}$$

involved in the third and the fifth terms of (3.4). In evaluation of  $H_{\mathbf{x}_i}$  and  $H_{b_i}$  we use the equalities

$$H_{\mathbf{x}_i}(\mathbf{x}_1, b_1, \mathbf{p}_1, q_1, \dots, \mathbf{x}_N, b_N, \mathbf{p}_N, q_N) = -L_{\mathbf{x}_i}(\mathbf{x}_1, b_1, \mathbf{v}_1, s_1, \dots, \mathbf{x}_N, b_N, \mathbf{v}_N, s_N),$$

$$H_{b_i}(\mathbf{x}_1, b_1, \mathbf{p}_1, q_1, \dots, \mathbf{x}_N, b_N, \mathbf{p}_N, q_N) = -L_{b_i}(\mathbf{x}_1, b_1, \mathbf{v}_1, s_1, \dots, \mathbf{x}_N, b_N, \mathbf{v}_N, s_N).$$

Here  $L$  is the Lagrangian and  $H$  is the Hamiltonian.

Integrating the expression for  $-L_{\mathbf{x}_1} f^{(N)}$  with respect to variables with indices  $j \neq 1$ , we get

$$\int H_{\mathbf{x}_1} f^{(N)} d\Omega_{N-1,1} = \rho \left( 4\pi b_1^2 \tilde{s}_1 \nabla\Phi + 2\pi b_1^3 \left( \frac{\partial}{\partial \mathbf{x}_1} \left( \nabla\Phi + \frac{1}{2} \int \tau \tilde{\mathbf{v}} f d\Omega \right) \right)^T \tilde{\mathbf{v}}_1 \right) f_1^{(1)}. \quad (3.11)$$

Analogous calculations give the following expression for

$$\int H_{b_1} f^{(N)} d\Omega_{N-1,1} = 4\pi b_1^2 \left( \rho \left( -\frac{\tilde{\mathbf{v}}_1^2}{4} - \frac{3\tilde{s}_1^2}{2} - \frac{3\tilde{\mathbf{v}}_1}{3\alpha_1 - 2} \cdot \left( \nabla\Phi + \frac{1}{\rho} \int \mathbf{p} f d\Omega \right) + \frac{2\tilde{s}_1\Phi}{b_1} \right) + \frac{d\varepsilon}{d\tau}(\tau_1) \right) f_1^{(1)}. \quad (3.12)$$

It follows from the formulae above that we can introduce the averaged Hamiltonian  $\tilde{H}(\mathbf{x}_1, b_1, \mathbf{p}_1, q_1)$  by the relations

$$\tilde{H}_{\mathbf{p}_1} = \tilde{\mathbf{v}}_1, \quad \tilde{H}_{q_1} = \tilde{s}_1, \quad \tilde{H}_{\mathbf{x}_1} f_1^{(1)} = \int H_{\mathbf{x}_1} f^{(N)} d\Omega_{N-1,1}, \quad \tilde{H}_{b_1} f_1^{(1)} = \int H_{b_1} f^{(N)} d\Omega_{N-1,1}. \quad (3.13)$$

It can be verified that the overdetermined system satisfies the compatibility conditions by virtue of (3.10)–(3.12). As a result, we derived the kinetic system governing the evolution of the density function in phase space (in the following we omit the tilde and the subscript “1”):

$$f_t + \operatorname{div}_{\mathbf{x}}(H_{\mathbf{p}} f) + (H_q f)_b - \operatorname{div}_{\mathbf{p}}(H_{\mathbf{x}} f) - (H_b f)_q = 0, \quad (3.14)$$

$$H(\mathbf{x}, b, \mathbf{p}, q) = \frac{\rho\tau}{4} \mathbf{v}^2 + \frac{3\rho\tau}{2} s^2 + \varepsilon(\tau), \quad (3.15)$$

$$\Delta\Phi = 4\pi \int b^2 s f d\Omega - \frac{3}{2} \operatorname{div} \left( \int \tau \mathbf{v} f d\Omega \right), \quad (3.16)$$

where

$$\mathbf{v} = \frac{2\mathbf{p}}{\rho\tau} - \frac{6}{3\alpha - 2} \left( \nabla\Phi + \frac{1}{\rho} \int \mathbf{p} f d\Omega \right), \quad s = \frac{q}{3\rho\tau} + \frac{\Phi}{b}, \quad \alpha = \int \tau f d\Omega. \quad (3.17)$$

Equation determining  $\Phi$  can also be written in the form

$$\Delta\Phi = \operatorname{div} \left( \frac{6}{\rho(3\alpha - 2)} \int \mathbf{p} f d\Omega + \frac{9\alpha}{3\alpha - 2} \nabla\Phi \right) + 4\pi \Phi \int b f d\Omega + \frac{1}{\rho} \int \frac{q}{b} f d\Omega.$$

It is interesting to note that  $H$  formally looks like the total energy of a single compressible bubble moving in a fluid.

Equations obtained can be used for the derivation of approximate models. If we write the equations in dimensionless variables

$$\mathbf{p} = \rho \bar{b}^3 V \mathbf{p}', \quad q = \rho \bar{b}^3 V q', \quad \mathbf{v} = V \mathbf{v}', \quad s = V s', \quad \Phi = \frac{\bar{b}^2}{d} V \Phi',$$

$$f = \frac{1}{\rho^4 \bar{b}^{13} d^3 V^4} f', \quad b = \bar{b} b', \quad \mathbf{x} = d \mathbf{x}', \quad t = \bar{b} V^{-1} t', \quad V = \left( \frac{P_\infty}{\rho} \right)^{1/2} \quad (3.18)$$



then the powers of the small parameter  $\beta = \bar{b}/d$  appear in the governing equations. Neglecting terms of order  $O(\beta^4)$  and higher, we get an approximate kinetic system (3.14)–(3.16) supplemented by the equations

$$\mathbf{v} = \frac{2\mathbf{p}}{\rho\tau} + 3\left(\nabla\Phi + \frac{1}{\rho} \int \mathbf{p}f \, d\Omega\right), \quad s = \frac{q}{3\rho\tau} + \frac{\Phi}{b}, \quad \alpha = \int \tau f \, d\Omega. \quad (3.19)$$

In (3.19) we have returned to dimensional variables. If we additionally drop terms of order  $O(\beta^4)$  and higher in expression (3.15) for the Hamiltonian, we obtain an analogue of Russo and Smereka system [22].

Dropping some  $O(\beta^3)$  terms in kinetic equations, we obtain another approximate model of the form (3.14)–(3.16) with Eqs. (3.17) replaced by

$$\mathbf{v} = \frac{2\mathbf{p}}{\rho\tau} + 3\nabla\Phi, \quad s = \frac{q}{3\rho\tau} + \frac{\Phi}{b}. \quad (3.20)$$

This system is similar to the kinetic equations derived in the case of rigid balls by Herrero et al. [11]. If we formally put  $f_b = f_q = 0$ ,  $s = 0$  and  $b = \text{const}$  in (3.14)–(3.16) and (3.18), we arrive at the kinetic model for the case of rigid balls [11].

#### 4. Ambient liquid flow

Description obtained admits a simple interpretation. A potential of the mean liquid flow induced by distant monopoles and dipoles is equal to  $\Phi$ . The gradient of  $\Phi$  gives the velocity induced by distant singularities

$$\mathbf{u}_0 = \nabla\Phi, \quad \text{div } \mathbf{u}_0 = 4\pi \int b^2 s f \, d\Omega - \frac{3}{2} \text{div} \left( \int \tau \mathbf{v} f \, d\Omega \right). \quad (4.1)$$

We see that the flow potential  $\Phi$  represents the averaged linear combination of potentials related to monopoles and dipoles. We can find the mean pressure by using the Cauchy–Lagrange integral

$$p = P_\infty - \rho \left( \Phi_t + \frac{1}{2} (\nabla\Phi)^2 \right). \quad (4.2)$$

From (4.1), (4.2) it follows that the flow induced by distant monopoles and dipoles is governed by irrotational Euler equations with source terms:

$$\begin{aligned} \rho(\mathbf{u}_{0t} + (\mathbf{u}_0 \cdot \nabla)\mathbf{u}_0) + \nabla p &= 0, \\ \text{div } \mathbf{u}_0 &= 4\pi \int b^2 s f \, d\Omega - \frac{3}{2} \text{div} \left( \int \tau \mathbf{v} f \, d\Omega \right), \quad \text{rot } \mathbf{u}_0 = 0. \end{aligned} \quad (4.3)$$

Analogously through the averaging of expression for gradient of potentials related to monopoles and dipoles we calculate the mean velocity of the liquid  $\mathbf{u}_l$ :

$$\mathbf{u}_l = \mathbf{u}_0 + \frac{1}{2} \int \tau \mathbf{v} f \, d\Omega.$$

In derivation of this expression formula (3.7) was used. The first term in the right-hand side of this equality describes contribution to the velocity field from distant monopoles and dipoles, and the second term represents the contribution induced by local dipoles. The bubble impulse

$$\mathbf{p} = \frac{\rho\tau}{2}(\mathbf{v} - 3\mathbf{u}_l) = \frac{\rho\tau}{2}(\mathbf{v} - \mathbf{u}_l) - \rho\tau\mathbf{u}_l \quad (4.4)$$

is the product of the added mass  $\rho\tau/2$  and the velocity  $\mathbf{v} - 3\mathbf{u}_l$ . It is known (see, for example, van Wijngaarden [31]) that a massless sphere moves, in an impulsively generated flow, with three times the velocity that this flow has at the centre of the sphere. Formula (4.4) agrees with this result when  $\mathbf{p} = 0$ . From the other hand, (4.4) shows that the bubble's impulse is equal to the difference between relative impulse (which is defined with respect to relative velocity  $\mathbf{v} - \mathbf{u}_l$ ) and momentum of the liquid in a volume  $\tau$ .

#### 5. Conservation laws for kinetic system

In this section we obtain the continuity equations for the liquid and gas phases, the conservation laws for total momentum and energy as a consequence of kinetic equation (3.14).

Integrating (3.14) with respect to  $\mathbf{p}$ ,  $q$ ,  $b$  and taking into account that  $f$  vanishes at infinity, we obtain the conservation law for the bubble number density:

$$n_t + \operatorname{div} \left( \int H_{\mathbf{p}} f \, d\Omega \right) = 0. \quad (5.1)$$

Here  $n = \int f \, d\Omega$  is the number of the bubbles per unit volume. Next conservation law is obtained after integration of (3.14) multiplied by  $4/3\pi b^3$

$$\left( \int \frac{4}{3}\pi b^3 f \, d\Omega \right)_t + \operatorname{div} \left( \int \frac{4}{3}\pi b^3 f H_{\mathbf{p}} \, d\Omega \right) = \int 4\pi b^2 H_q f \, d\Omega = \int 4\pi b^2 s f \, d\Omega = \Delta \varphi. \quad (5.2)$$

Taking into account the equality

$$\operatorname{div} \left( \nabla \phi + \frac{3}{2} \int \tau \mathbf{v} f \, d\Omega \right) = 0$$

(see (3.8)) we derive the equation

$$\alpha_t + \operatorname{div} \left( \int \tau \mathbf{v} f \, d\Omega \right) = \operatorname{div} \left( \mathbf{u}_0 + \frac{3}{2} \int \tau \mathbf{v} f \, d\Omega \right). \quad (5.3)$$

It is natural to define the mean mass velocity  $\mathbf{U}$  of the liquid phase:

$$\mathbf{U} = (1 - \alpha)^{-1} \mathbf{u}_l.$$

Eq. (5.3) can be rewritten as a conservation law for a mass of the liquid

$$(\rho(1 - \alpha))_t + \operatorname{div}(\rho(1 - \alpha)\mathbf{U}) = 0 \quad (5.4)$$

flowing with the mean velocity  $\mathbf{U}$ .

To obtain the conservation law for momentum, we multiply Eq. (3.14) by  $p_i$  and integrate with respect to variables  $\mathbf{p}$ ,  $b$ ,  $q$ :

$$\left( \int p_i f \, d\Omega \right)_t + \operatorname{div} \left( \int p_i H_{\mathbf{p}} f \, d\Omega \right) + \int H_{x_i} f \, d\Omega = 0, \quad i = 1, 2, 3.$$

Here  $\mathbf{p} = (p_1, p_2, p_3)^T$ ,  $\mathbf{x} = (x_1, x_2, x_3)^T$ . Taking into account that

$$\begin{aligned} \int H_{x_i} f \, d\Omega &= \operatorname{div} \left( \frac{3}{2} \rho \Phi_{x_i} \int \tau \mathbf{v} f \, d\Omega \right) + \rho \Delta \Phi \Phi_{x_i} + \frac{3}{8} \rho \left( \int \tau \mathbf{v} f \, d\Omega \right)_{x_i}^2, \\ \Delta \Phi \Phi_{x_i} &= \operatorname{div}(\Phi_{x_i} \nabla \Phi) - \frac{1}{2} (\nabla \Phi)_{x_i}^2, \end{aligned} \quad (5.5)$$

we obtain the conservation law for the total momentum of the bubbles:

$$\left( \int \mathbf{p} f \, d\Omega \right)_t + \operatorname{div} \left( \int (\mathbf{p} \otimes H_{\mathbf{p}}) f \, d\Omega + \rho \mathbf{u}_0 \otimes \left( \mathbf{u}_0 + \frac{3}{2} \int \tau \mathbf{v} f \, d\Omega \right) \right) + \nabla \left( \frac{3}{8} \rho \left( \int \tau \mathbf{v} f \, d\Omega \right)^2 - \frac{\rho}{2} \mathbf{u}_0^2 \right) = 0. \quad (5.6)$$

Here the divergence of a second-order tensor  $A$  is defined as [21]

$$\operatorname{div}(A)\mathbf{a} = \operatorname{div}(A^T \mathbf{a})$$

for any constant vector  $\mathbf{a}$ . In particular,

$$\operatorname{div}(\mathbf{a} \otimes \mathbf{b}) = \mathbf{a} \operatorname{div} \mathbf{b} + \frac{\partial \mathbf{a}}{\partial \mathbf{x}} \mathbf{b}$$

for any vector fields  $\mathbf{a}$  and  $\mathbf{b}$ . We note that if a different definition is employed [26]:  $\operatorname{div}(A)\mathbf{a} = \operatorname{div}(A\mathbf{a})$ , it is sufficient to transpose the tensor product to obtain a right expression. Summing Eq. (5.6) with (4.4) written in a conservation form:

$$\rho \mathbf{u}_{0t} + \nabla \left( p + \frac{\rho}{2} \mathbf{u}_0^2 \right) = 0,$$

we obtain the conservation law for the total momentum of translational motion of gas–liquid system

$$\left( \int \mathbf{p} f \, d\Omega + \rho \mathbf{u}_0 \right)_t + \operatorname{div} \left( \int (\mathbf{p} \otimes H \mathbf{p}) f \, d\Omega + \rho \mathbf{u}_0 \otimes \left( \mathbf{u}_0 + \frac{3}{2} \int \tau \mathbf{v} f \, d\Omega \right) \right) + \nabla \left( p + \frac{3}{8} \rho \left( \int \tau \mathbf{v} f \, d\Omega \right)^2 \right) = 0. \quad (5.7)$$

In the following  $\mathbf{V}$  stands for mean value of the bubble's velocity

$$\mathbf{V} = \alpha^{-1} \int \tau \mathbf{v} f \, d\Omega.$$

Using the identity

$$\int \tau (\mathbf{v} \otimes \mathbf{v}) f \, d\Omega = \alpha \mathbf{V} \otimes \mathbf{V} + \int \tau (\mathbf{v} - \mathbf{V}) \otimes (\mathbf{v} - \mathbf{V}) f \, d\Omega$$

we rewrite conservation law (5.7) in the following form:

$$\rho \left( \left( 1 - \frac{3\alpha}{2} \right) (1 - \alpha) \mathbf{U} \right)_t + \operatorname{div} \left( \rho \left( 1 - \frac{3\alpha}{2} \right) (1 - \alpha) \mathbf{U} \otimes \mathbf{U} + \frac{\rho \alpha (1 - \alpha)}{2} (\mathbf{U} - \mathbf{V}) \otimes (\mathbf{U} - \mathbf{V}) \right) + \nabla \left( p + \frac{3}{8} \rho \alpha^2 \mathbf{V}^2 \right) = - \operatorname{div} \int \frac{\rho \tau}{2} (\mathbf{v} - \mathbf{V}) \otimes (\mathbf{v} - \mathbf{V}) f \, d\Omega. \quad (5.8)$$

Similarly we obtain inhomogeneous conservation law for the momentum of dilatational bubble motion

$$\left( \int q f \, d\Omega \right)_t + \operatorname{div} \left( \int q H \mathbf{p} f \, d\Omega \right) + \int H_b f \, d\Omega = 0.$$

The energy conservation law can be obtained from Eq. (3.14) if we multiply it by  $H$  and integrate with respect to variables  $\mathbf{p}$ ,  $q$ ,  $b$ :

$$\left( \int H f \, d\Omega \right)_t + \operatorname{div} \left( \int H H \mathbf{p} f \, d\Omega \right) - \int H_t f \, d\Omega = 0.$$

To transform to a divergent form, we use formulae similar to (5.5) and arrive at equation

$$\left( \int H f \, d\Omega + \frac{\rho}{2} \mathbf{u}_0^2 - \frac{3}{8} \left( \int \tau \mathbf{v} f \, d\Omega \right)^2 \right)_t + \operatorname{div} \left( \int H H \mathbf{p} f \, d\Omega + \left( p - P_\infty + \rho \frac{\mathbf{u}_0^2}{2} \right) \left( \mathbf{u}_0 + \frac{3}{2} \int \tau \mathbf{v} f \, d\Omega \right) \right) = 0.$$

Let us define the internal energy of bubbly liquid per unit volume  $E$  by the relation

$$E = \frac{1}{\alpha} \left( \frac{3\rho}{2} \int \tau s^2 f \, d\Omega + \int \varepsilon(\tau) f \, d\Omega + \frac{\rho \alpha (1 - \alpha)}{4} (\mathbf{U} - \mathbf{V})^2 + \frac{\rho}{4} \int \tau (\mathbf{v} - \mathbf{V})^2 f \, d\Omega \right). \quad (5.9)$$

Here the two first terms represent the pulsation energy and the internal energy of bubbles. The third term gives the energy of relative motion, and the last term corresponds to energy of bubble velocity fluctuations. Using the identities

$$\int \tau \mathbf{v}^2 f \, d\Omega = \alpha \mathbf{V}^2 + \int \tau (\mathbf{v} - \mathbf{V})^2 f \, d\Omega, \quad \frac{\alpha}{4} \mathbf{V}^2 + \frac{1}{2} \mathbf{u}_0^2 - \frac{3}{8} \alpha^2 \mathbf{V}^2 = \frac{1 - \alpha}{4} (\alpha (\mathbf{U} - \mathbf{V})^2 + (2 - 3\alpha) \mathbf{U}^2),$$

we write the energy conservation law in the form

$$\begin{aligned} & \left( \alpha E + \frac{\rho}{2} (1 - \alpha) \left( 1 - \frac{3\alpha}{2} \right) \mathbf{U}^2 \right)_t + \operatorname{div} \left( \left( \alpha E + \frac{\rho}{2} (1 - \alpha) \left( 1 - \frac{3\alpha}{2} \right) \mathbf{U}^2 + p + \frac{3}{8} \rho \alpha^2 \mathbf{V}^2 \right) \mathbf{V} \right. \\ & \quad \left. + (1 - \alpha) \left( p - P_\infty + \frac{1}{2} \rho \left( (1 - \alpha) \mathbf{U} - \frac{\alpha}{2} \mathbf{V} \right)^2 \right) (\mathbf{U} - \mathbf{V}) \right) \\ & = - \operatorname{div} \int \tau (H_1 - H_2) (\mathbf{v} - \mathbf{V}) f \, d\Omega. \end{aligned} \quad (5.10)$$

Here  $H_1 = \tau^{-1} H$ ,  $H_2 = \alpha^{-1} \int \tau H f \, d\Omega$ .

In the conservation laws derived the left-hand sides represent the terms related to the mean values of flow parameters. The terms in the right-hand side and the term in expression (5.9) for internal energy describe the contributions associated with the fluctuations of flow parameters. These terms can be expressed through the mean parameters if we chose some special form of the distribution function.

## 6. Hydrodynamical equations

For smoothly varying flows the distribution function is narrowly peaked. We shall derive a closed system of hydrodynamical equations describing such flows. If instability will develop, we have to take into consideration fluctuation terms. Taking into account that for the distribution function of the form

$$f = n(t, \mathbf{x}) \delta(\mathbf{p} - \mathbf{p}(t, \mathbf{x})) \delta(q - q(t, \mathbf{x})) \delta(b - b(t, \mathbf{x})) \quad (6.1)$$

the right-hand sides in (5.8), (5.10) vanish, we obtain the following conservation laws from (5.1), (5.4), (5.8) and (5.10):

$$\begin{aligned} n_t + \operatorname{div}(n\mathbf{V}) &= 0, & (\rho(1-\alpha))_t + \operatorname{div}(\rho(1-\alpha)\mathbf{U}) &= 0, \\ \rho \left( \left( 1 - \frac{3\alpha}{2} \right) (1-\alpha)\mathbf{U} \right)_t + \operatorname{div} \left( \rho \left( 1 - \frac{3\alpha}{2} \right) (1-\alpha)\mathbf{U} \otimes \mathbf{U} + \frac{\rho\alpha(1-\alpha)}{2} (\mathbf{U} - \mathbf{V}) \otimes (\mathbf{U} - \mathbf{V}) \right) \\ &+ \nabla \left( p + \frac{3}{8} \rho \alpha^2 \mathbf{V}^2 \right) &= 0, \\ \left( \alpha E + \frac{\rho}{2} (1-\alpha) \left( 1 - \frac{3\alpha}{2} \right) \mathbf{U}^2 \right)_t + \operatorname{div} \left( \left( \alpha E + \frac{\rho}{2} (1-\alpha) \left( 1 - \frac{3\alpha}{2} \right) \mathbf{U}^2 + p + \frac{3}{8} \rho \alpha^2 \mathbf{V}^2 \right) \mathbf{V} \right. \\ &+ (1-\alpha) \left( p - P_\infty + \frac{1}{2} \rho \left( (1-\alpha)\mathbf{U} - \frac{\alpha}{2} \mathbf{V} \right)^2 \right) (\mathbf{U} - \mathbf{V}) \Big) &= 0. \end{aligned} \quad (6.2)$$

Here  $\mathbf{U}$ ,  $\mathbf{V}$ ,  $\Phi$  and  $p$  are related by the equalities

$$(1-\alpha)\mathbf{U} - \frac{\alpha}{2}\mathbf{V} = \nabla\Phi, \quad p + \frac{\rho}{2}(\nabla\Phi)^2 = P_\infty - \rho\Phi_t. \quad (6.3)$$

System (6.2) complemented by the equations

$$\rho \left( (1-\alpha)\mathbf{U} - \frac{\alpha}{2}\mathbf{V} \right)_t + \nabla \left( p + \frac{\rho}{2} \left( (1-\alpha)\mathbf{U} - \frac{\alpha}{2}\mathbf{V} \right)^2 \right) = 0, \quad \operatorname{rot} \left( (1-\alpha)\mathbf{U} - \frac{\alpha}{2}\mathbf{V} \right) \Big|_{t=0} = 0 \quad (6.4)$$

form a closed system of hydrodynamical equations governing motions of the bubbles in an ideal liquid. The last condition prescribing irrotational initial data provides the equivalence of (6.3) and (6.4). Substituting (6.1) into the equation

$$\alpha_t + \operatorname{div}(\alpha\mathbf{V}) = 4\pi \int b^2 s f \, d\Omega$$

we derive the relation

$$s = \dot{b} = b_t + (\mathbf{V} \cdot \nabla)b.$$

The Galilean invariant part of the energy

$$E = \frac{3}{2} \rho (\dot{b})^2 + e(\tau) + \frac{\rho(1-\alpha)}{4} (\mathbf{U} - \mathbf{V})^2 \quad (6.5)$$

depends not only on  $b$ , but also on  $\dot{b}$  and  $(\mathbf{U} - \mathbf{V})^2$ . We introduced here the notation  $e(\tau)$  for  $\tau^{-1}\varepsilon(\tau)$ . In a system (6.2), (6.4) consisting of two vector equations and three scalar equations there are two unknown vectors  $\mathbf{U}$ ,  $\mathbf{V}$  and three scalar unknown functions  $b$ ,  $\alpha$ ,  $p$ .

**Remark.** We see that some coefficients in Eqs. (6.2), (6.4) vanish at  $\alpha = 2/3$ . One can expect that the solutions of the system have singular behavior when  $\alpha$  approaches  $2/3$ . The total energy of the system

$$\alpha E + \frac{\rho}{2} (1-\alpha) \left( 1 - \frac{3\alpha}{2} \right) \mathbf{U}^2$$

can be negative for  $\alpha > 2/3$ . This values of gas volume concentration do not belong to the range of our model validation. The value  $\alpha = 2/3$  is very close to the limiting volume concentration for a random dense packing of spheres in a volume.

## 7. Rayleigh–Lamb’s equation

The oscillations of the bubbles are governed by Rayleigh–Lamb’s equation which is a consequence of conservation laws (6.2). To show this fact we transform the momentum conservation law to the form:

$$\begin{aligned} \frac{\rho\alpha}{2} \left(1 - \frac{3\alpha}{2}\right) (\mathbf{V}_t + (\mathbf{V} \cdot \nabla) \mathbf{V}) + \frac{3\rho\alpha^2}{4} \left(\frac{\partial \mathbf{V}}{\partial \mathbf{x}}\right)^T \mathbf{V} - \frac{3\rho\alpha}{4} \mathbf{V} \operatorname{div} \left(\mathbf{u}_0 + \frac{\alpha}{2} \mathbf{V}\right) + \frac{3\rho\alpha}{4} (\mathbf{V}^2 \nabla \alpha - \mathbf{V} (\mathbf{V} \cdot \nabla) \alpha) \\ + \frac{3\alpha}{2} \nabla p + \frac{3\rho\alpha}{2} (\mathbf{u}_0 \cdot \nabla) \mathbf{u}_0 + 2\pi\rho b^2 sn \left( \left(1 - \frac{3\alpha}{2}\right) \mathbf{V} - \mathbf{u}_0 \right) = 0. \end{aligned} \quad (7.1)$$

Transforming the energy conservation law we obtain the equation:

$$\begin{aligned} 3\alpha\rho s \dot{s} + 4\pi b^2 sn \left( \frac{\rho}{4} \mathbf{V}^2 + \frac{3}{2} \rho s^2 + \varepsilon' + p - P_\infty + \frac{\rho}{2} \mathbf{u}_0^2 \right) + \frac{\rho\alpha}{2} \left(1 - \frac{3\alpha}{2}\right) \mathbf{V} \cdot \mathbf{V}_t + \frac{\rho\alpha}{2} \mathbf{V} \cdot ((\mathbf{V} \cdot \nabla) \mathbf{V}) \\ + \frac{3\alpha}{2} (\mathbf{V} \cdot \nabla) \left( p - P_\infty + \frac{\rho}{2} \mathbf{u}_0^2 \right) - \frac{3\rho\alpha}{4} \mathbf{V}^2 \operatorname{div} \left( \mathbf{u}_0 + \frac{\alpha}{2} \mathbf{V} \right) = 0. \end{aligned} \quad (7.2)$$

Here  $\dot{s} = s_t + (\mathbf{V} \cdot \nabla) s$ . Summing Eq. (7.2) with (7.1) multiplied by  $-\mathbf{V}$  we obtain Rayleigh–Lamb’s equation

$$b\dot{s} + \frac{3}{2} s^2 + \frac{p - p_g}{\rho} + \frac{1}{2} \left( \mathbf{u}_0 + \frac{1}{2} \mathbf{V} \right)^2 - \frac{3(1 - 2\alpha)}{8} \mathbf{V}^2 = 0. \quad (7.3)$$

Eq. (7.3) allows us to find  $p$  as a function of  $b$ ,  $s$ ,  $\dot{s}$ ,  $n$ ,  $\mathbf{U}$  and  $\mathbf{V}$ . In a new formulation we obtain the system consisting of the first three Eqs. (6.2), (6.4) and Rayleigh–Lamb’s equation (7.3) for the pressure. Notice, that Rayleigh–Lamb’s equation follows immediately from the momentum equation for dilatational bubble motion. Let us rewrite Rayleigh–Lamb’s equation in the following equivalent form:

$$b\dot{s} + \frac{3}{2} s^2 + \frac{p - p_g}{\rho} - \frac{(1 - \alpha)^2}{4} (\mathbf{U} - \mathbf{V})^2 + \frac{3(1 - \alpha)^2}{4} \mathbf{U}^2 + \frac{3\alpha^2}{8} \mathbf{V}^2 = 0. \quad (7.4)$$

To estimate the asymptotic accuracy of (7.4), we notice that

$$\mathbf{u}_0 = \nabla \Phi = (1 - \alpha) \mathbf{U} - \alpha/2 \mathbf{V}$$

is determined from the equation (see (4.1))

$$\Delta \Phi = 4\pi b^2 sn - \frac{3}{2} \operatorname{div}(\alpha \mathbf{V}) \quad (7.5)$$

and the condition of vanishing of  $\Phi$  at infinity. Taking into account that for  $sn^{1/3}$  of order  $O(1)$  the right-hand side of (7.5) is of order  $O(\alpha^{2/3})$ , we can conclude that  $\mathbf{u}_0$  and  $\mathbf{U}$  are also of order  $O(\alpha^{2/3})$ . It gives the asymptotic estimate  $O(\alpha^{4/3})$  for the difference between the left-hand sides of (7.4) and the classical Rayleigh–Lamb’s equation (see, for instance, Biesheuvel and van Wijngaarden [2]):

$$b\dot{s} + \frac{3}{2} s^2 + \frac{p - p_g}{\rho} - \frac{(1 - \alpha)^2}{4} (\mathbf{U} - \mathbf{V})^2 = 0.$$

But we have to keep the small-order terms to conserve Hamiltonian and Lagrangian structures of the system derived.

## 8. Variational structure of hydrodynamical system

A natural question appears whether hydrodynamical system (6.2), (6.4) has a variational structure. The answer is not obvious even if the initially formulated problem has this property. Indeed, several hypotheses were used for derivation of the governing equations. This, in principle, could change the initial structure of the system. Nevertheless, we shall show that the system can be obtained by using the Lagrangian formulation. In the following it is more convenient to consider the Lagrangian

$$L = \frac{\rho}{2} (1 - \alpha) \left(1 - \frac{3\alpha}{2}\right) \mathbf{U}^2 + \frac{\rho(1 - \alpha)\alpha}{4} (\mathbf{U} - \mathbf{V})^2 + \frac{3}{2} \rho \alpha (\dot{b})^2 - \alpha e(\tau) \quad (8.1)$$

as function of variables:

$$\rho_l = (1 - \alpha)\rho, \quad \rho_g = n, \quad \mathbf{j}_l = \rho_l \mathbf{U}, \quad \mathbf{j}_g = \rho_g \mathbf{V}, \quad \tau = \frac{4}{3} \pi b^3, \quad \dot{\tau} = \frac{\partial \tau}{\partial t} + \frac{\mathbf{j}_g}{\rho_g} \cdot \nabla \tau. \quad (8.2)$$

Notice, that  $n$  has no the dimension of density, but its product with the mass has. We can express the specific volume  $\tau$  in terms of  $\rho_l$  and  $\rho_g$ :

$$\tau = \frac{\alpha}{n} = \frac{\rho - \rho_l}{\rho \rho_g}. \quad (8.3)$$

The Lagrangian (8.1) considered as function of variables (8.2) is:

$$L(\rho_l, \rho_g, \mathbf{j}_l, \mathbf{j}_g, \tau, \dot{\tau}) = \left(1 - \frac{3\tau\rho_g}{2}\right) \frac{\mathbf{j}_l^2}{2\rho_l} + \frac{\rho_l\tau\rho_g}{4} \left(\frac{\mathbf{j}_l}{\rho_l} - \frac{\mathbf{j}_g}{\rho_g}\right)^2 + \frac{\rho\rho_g}{8\pi b} (\dot{\tau})^2 - \rho_g \varepsilon(\tau). \quad (8.4)$$

Substituting expressions for  $\tau$  and  $b$  following from (8.2), (8.3), we can write a general form of the Lagrangian:

$$\tilde{L}\left(\rho_l, \rho_g, \mathbf{j}_l, \mathbf{j}_g, \frac{\partial \rho_l}{\partial t}, \nabla \rho_l, \frac{\partial \rho_g}{\partial t}, \nabla \rho_g\right) = L(\rho_l, \rho_g, \mathbf{j}_l, \mathbf{j}_g, \tau, \dot{\tau}). \quad (8.5)$$

Next we obtain hydrodynamical system (6.2), (6.4) from Hamilton's principle of stationary action. This principle states that the Hamilton action

$$a = \int L \, dt \, d\mathbf{x}$$

is stationary with respect to arbitrary variations of fluid particle displacements satisfying the constraints

$$\frac{\partial \rho_\kappa}{\partial t} + \operatorname{div} \mathbf{j}_\kappa = 0, \quad \kappa = l, g. \quad (8.6)$$

Here the integral is taken over the whole volume of the fluid. We shall show that system (6.2), (6.4) determines "potential solutions" of the Euler–Lagrange equations written for the constrained Lagrangian (8.4) (the notion of "potentiality" will be defined below). We shall not use the conventional method of Lagrange multipliers. Instead of it we integrate the constraints in Lagrangian coordinates and solve them with respect to the densities (see, for example, Salmon [23] for Euler equations, Gavriluk and Gouin [8] for two-fluid equations).

For this, we consider two smooth one-parameter families of virtual motions:

$$\mathbf{x} = \Phi_\kappa(\mathbf{X}_\kappa, t, \lambda_\kappa), \quad \kappa = l, g,$$

where  $\mathbf{X}_\kappa$  are the Lagrangian coordinates of each component, and  $\lambda_\kappa$  are small parameters. It is supposed that  $\lambda_\kappa = 0$ , correspond to the real motion of the continua

$$\mathbf{x} = \varphi_\kappa(\mathbf{X}_\kappa, t), \quad \kappa = l, g.$$

The virtual displacements

$$\xi_\kappa(\mathbf{x}, t) = \frac{\partial \Phi_\kappa(\mathbf{X}_\kappa, t, 0)}{\partial \lambda_\kappa}, \quad \mathbf{X}_\kappa = \varphi_\kappa^{-1}(\mathbf{x}, t), \quad \kappa = l, g,$$

are supposed to be functions with compact support. Variations of dependent variables  $\rho_\kappa$  and  $\mathbf{j}_\kappa$  in Eulerian coordinates can be expressed in terms of  $\xi_\kappa$ :

$$\delta_\kappa \rho_\kappa = -\operatorname{div}(\rho_\kappa \xi_\kappa).$$

This formula can be derived from the mass conservation laws (8.6) integrated in Lagrangian coordinates

$$\rho_\kappa \det\left(\frac{\partial \mathbf{x}}{\partial \mathbf{X}_\kappa}\right) = \rho_{\kappa 0}(\mathbf{X}_\kappa), \quad \kappa = l, g.$$

The space variation of  $\mathbf{j}_\kappa$  admit the representation [8, Appendix B]:

$$\delta_\kappa \mathbf{j}_\kappa = \frac{\partial \rho_\kappa \xi_\kappa}{\partial t} + \operatorname{div}(\xi_\kappa \otimes \mathbf{j}_\kappa - \mathbf{j}_\kappa \otimes \xi_\kappa).$$

Taking into account that the virtual displacements  $\xi_\kappa$  vanish at the boundary and using the divergence theorem, we represent the variations of the Hamilton action as the linear functional over  $\xi_\kappa$ :

$$\begin{aligned} \delta_\kappa a &= \left. \frac{da}{d\lambda_\kappa} \right|_{\lambda_\kappa=0} = \delta_\kappa \int \tilde{L}\left(\rho_l, \rho_g, \mathbf{j}_l, \mathbf{j}_g, \frac{\partial \rho_l}{\partial t}, \nabla \rho_l, \frac{\partial \rho_g}{\partial t}, \nabla \rho_g\right) dt \, d\mathbf{x} = \int \left( \frac{\delta \tilde{L}}{\delta \rho_\kappa} \delta_\kappa \rho_\kappa + \frac{\partial \tilde{L}}{\partial \mathbf{j}_\kappa} \cdot \delta_\kappa \mathbf{j}_\kappa \right) dt \, d\mathbf{x} \\ &= \int (R_\kappa \delta_\kappa \rho_\kappa + \mathbf{K}_\kappa \delta_\kappa \mathbf{j}_\kappa) dt \, d\mathbf{x} = - \int \left( \rho_\kappa \frac{\partial \mathbf{K}_\kappa}{\partial t} + \operatorname{rot} \mathbf{K}_\kappa \wedge \mathbf{j}_\kappa - \rho_\kappa \nabla R_\kappa \right) \xi_\kappa dt \, d\mathbf{x} = 0. \end{aligned} \quad (8.7)$$

Here

$$\frac{\delta \tilde{L}}{\delta \rho_\kappa} = \frac{\partial \tilde{L}}{\partial \rho_\kappa} - \frac{\partial}{\partial t} \left( \frac{\partial \tilde{L}}{\partial (\partial \rho_\kappa / \partial t)} \right) - \operatorname{div} \left( \frac{\partial \tilde{L}}{\partial \nabla \rho_\kappa} \right), \quad R_\kappa = \frac{\delta \tilde{L}}{\delta \rho_\kappa}, \quad \mathbf{K}_\kappa = \frac{\partial \tilde{L}}{\partial \mathbf{j}_\kappa}. \quad (8.8)$$

From (8.7) by virtue of arbitrariness of  $\xi_\kappa$ , we get the equations of the motion:

$$\rho_\kappa \frac{\partial \mathbf{K}_\kappa}{\partial t} + \operatorname{rot} \mathbf{K}_\kappa \wedge \mathbf{j}_\kappa - \rho_\kappa \nabla R_\kappa = 0, \quad \kappa = l, g. \quad (8.9)$$

In the case of irrotational (potential) motions which are defined by the relation (see Gavriluyk and Gouin [8] and Gavriluyk and Teshukov [9])

$$\operatorname{rot} \mathbf{K}_\kappa = 0,$$

Eqs. (8.9) are written in conservative form. In general case only the total momentum and the total energy are conserved (Noether's theorem). Summing (8.9) we obtain the conservation law for total momentum:

$$\begin{aligned} \frac{\partial}{\partial t} \left( \sum_{\kappa=l,g} \left( \rho_\kappa \mathbf{K}_\kappa - \frac{\partial \tilde{L}}{\partial (\partial \rho_\kappa / \partial t)} \nabla \rho_\kappa \right) \right) + \operatorname{div} \left( \sum_{\kappa=l,g} \left( \mathbf{K}_\kappa \otimes \mathbf{j}_\kappa - \nabla \rho_\kappa \otimes \frac{\partial \tilde{L}}{\partial \nabla \rho_\kappa} \right) \right) \\ + \left( \tilde{L} - \sum_{\kappa=l,g} (\rho_\kappa R_\kappa + \mathbf{K}_\kappa \cdot \mathbf{j}_\kappa) \right) \mathbf{I} = 0. \end{aligned} \quad (8.10)$$

Linear combination of Eqs. (8.9) with multiplies  $\mathbf{U}$  and  $\mathbf{V}$  gives the conservation law for total energy:

$$\frac{\partial}{\partial t} \left( \sum_{\kappa=l,g} \left( \mathbf{K}_\kappa \cdot \mathbf{j}_\kappa + \frac{\partial \rho_\kappa}{\partial t} \frac{\partial \tilde{L}}{\partial (\partial \rho_\kappa / \partial t)} \right) - \tilde{L} \right) + \operatorname{div} \left( \sum_{\kappa=l,g} \left( -R_\kappa \mathbf{j}_\kappa + \frac{\partial \rho_\kappa}{\partial t} \frac{\partial \tilde{L}}{\partial \nabla \rho_\kappa} \right) \right) = 0. \quad (8.11)$$

Let us write the Euler–Lagrange equations for the Lagrangian (8.4) in explicit form. Evaluating

$$\mathbf{K}_l = \frac{\partial \tilde{L}}{\partial \mathbf{j}_l} = \frac{\partial L}{\partial \mathbf{j}_l} = \left( 1 - \frac{3\alpha}{2} \right) \frac{\mathbf{j}_l}{\rho_l} + \frac{\alpha}{2} \left( \frac{\mathbf{j}_l}{\rho_l} - \frac{\mathbf{j}_g}{\rho_g} \right) = (1 - \alpha) \mathbf{U} - \frac{\alpha}{2} \mathbf{V} = \mathbf{u}_0$$

we see that the velocity conjugated to the fluid momentum  $\mathbf{j}_l$  coincides with the induced fluid velocity  $\mathbf{u}_0$ . Notice that if  $\mathbf{u}_0$  is irrotational then the momentum equation (8.8) for  $\kappa = l$  is in conservative form. Let us show that in this case  $R_l$  is equal to

$$-\left( \frac{p - P_\infty}{\rho} + \frac{\mathbf{u}_0^2}{2} \right).$$

Indeed, referring to (8.4), (8.5) we get

$$R_l = \frac{\delta \tilde{L}}{\delta \rho_l} = \frac{\partial L}{\partial \rho_l} + \frac{\delta L}{\delta \tau} \frac{\partial \tau}{\partial \rho_l} = \frac{\partial L}{\partial \rho_l} - \frac{1}{n\rho} \frac{\delta L}{\delta \tau}. \quad (8.12)$$

Evaluating the partial derivative of  $L(\rho_l, \rho_g, \mathbf{j}_l, \mathbf{j}_g, \tau, \dot{\tau})$ ,

$$\frac{\partial L}{\partial \rho_l} = -\left( 1 - \frac{3\alpha}{2} \right) \frac{\mathbf{j}_l^2}{2\rho_l^2} + \frac{n\tau}{4} \left( \frac{\mathbf{j}_l}{\rho_l} - \frac{\mathbf{j}_g}{\rho_g} \right)^2 + \frac{\rho_l \alpha}{2} \left( \frac{\mathbf{j}_l}{\rho_l} - \frac{\mathbf{j}_g}{\rho_g} \right) \left( -\frac{\mathbf{j}_l}{\rho_l^2} \right) = \frac{\alpha}{4} \mathbf{V}^2 - \frac{(1-\alpha)}{2} \mathbf{U}^2,$$

and the term

$$\begin{aligned} -\frac{1}{n\rho} \frac{\delta L}{\delta \tau} &= -\frac{1}{n\rho} \left( -\frac{3n}{4\rho_l} \mathbf{j}_l^2 + \frac{n\rho_l}{4} \left( \frac{\mathbf{j}_l}{\rho_l} - \frac{\mathbf{j}_g}{\rho_g} \right)^2 - \frac{n\rho_l \dot{\tau}^2}{8\pi b^2} \frac{1}{4\pi b^2} - n \frac{d\varepsilon}{d\tau} - \frac{\partial}{\partial t} \left( \frac{n\rho \dot{\tau}}{4\pi b} \right) - \operatorname{div} \left( \frac{n\rho \dot{\tau}}{4\pi b} \mathbf{V} \right) \right) \\ &= \frac{3(1-\alpha)}{4} \mathbf{U}^2 - \frac{(1-\alpha)}{4} (\mathbf{U} - \mathbf{V})^2 + \frac{1}{\rho} \frac{d\varepsilon}{d\tau} + b\ddot{b} + \frac{3}{2} \dot{b}^2, \end{aligned}$$

we arrive at the final expression for

$$\begin{aligned} R_l &= \frac{\alpha}{4} \mathbf{V}^2 - \frac{(1-\alpha)}{2} \mathbf{U}^2 + \frac{3(1-\alpha)}{4} \mathbf{U}^2 - \frac{(1-\alpha)}{4} (\mathbf{U} - \mathbf{V})^2 + \frac{1}{\rho} \frac{d\varepsilon}{d\tau} + b\ddot{b} + \frac{3}{2} \dot{b}^2 \\ &= \frac{\alpha}{4} \mathbf{V}^2 + \frac{(1-\alpha)}{4} \mathbf{U}^2 - \frac{(1-\alpha)}{4} (\mathbf{U} - \mathbf{V})^2 + \frac{1}{\rho} \frac{d\varepsilon}{d\tau} + b\ddot{b} + \frac{3}{2} \dot{b}^2. \end{aligned}$$

If we prove that

$$R_l + \frac{\mathbf{u}_0^2}{2} = \frac{P_\infty - p}{\rho} \quad (8.13)$$

then the equation

$$\frac{\partial \mathbf{K}_l}{\partial t} - \nabla R_l = 0 \quad (8.14)$$

is equivalent to (6.4). It is easy to see that (8.13) is equivalent to Rayleigh–Lamb’s equation (7.3). This proves the equivalence of (6.4) and (8.14).

Taking into account that the total momentum

$$\sum_{\kappa=l,g} \left( \rho_\kappa \mathbf{K}_\kappa - \frac{\partial \tilde{L}}{\partial(\partial \rho_\kappa / \partial t)} \nabla \rho_\kappa \right)$$

is equal to

$$\begin{aligned} \rho_l \frac{\partial L}{\partial \mathbf{j}_l} + n \frac{\partial L}{\partial \mathbf{j}_g} + \frac{\partial L}{\partial \tau} \nabla \tau - \frac{\partial L}{\partial \dot{\tau}} \left( \frac{\partial \tau}{\partial \rho_l} \nabla \rho_l + \frac{\partial \tau}{\partial n} \nabla n \right) &= \rho_l \frac{\partial L}{\partial \mathbf{j}_l} + n \frac{\partial L}{\partial \mathbf{j}_g} = \rho_l (1 - \alpha) \mathbf{U} - \frac{\alpha \rho_l}{2} \mathbf{V} - \frac{\alpha \rho_l}{2} (\mathbf{U} - \mathbf{V}) \\ &= \rho (1 - \alpha) \left( 1 - \frac{3\alpha}{2} \right) \mathbf{U}, \end{aligned}$$

we see that the conserved quantity in (8.10) is the total momentum presenting in the system (6.2). Notice that the total momentum does not depend on  $\mathbf{V}$ . Next we show that the flux

$$\left( \mathbf{K}_\kappa \otimes \mathbf{j}_\kappa - \nabla \rho_\kappa \otimes \frac{\partial \tilde{L}}{\partial \nabla \rho_\kappa} \right) + \left( \tilde{L} - \sum_{\kappa=l,g} (\rho_\kappa R_\kappa + \mathbf{K}_\kappa \cdot \mathbf{j}_\kappa) \right) I \quad (8.15)$$

coincides with the flux

$$\rho \left( 1 - \frac{3\alpha}{2} \right) (1 - \alpha) \mathbf{U} \otimes \mathbf{U} + \frac{\rho \alpha (1 - \alpha)}{2} (\mathbf{U} - \mathbf{V}) \otimes (\mathbf{U} - \mathbf{V}) + \left( p - P_\infty + \frac{3}{8} \rho \alpha^2 \mathbf{V}^2 \right) I \quad (8.16)$$

of the third equation of system (6.2). To prove this, we compare independently the dyad terms and diagonal tensors. We have:

$$\begin{aligned} \mathbf{K}_l &= \frac{\partial \tilde{L}}{\partial \mathbf{j}_l} = \frac{\partial L}{\partial \mathbf{j}_l} = (1 - \alpha) \mathbf{U} - \frac{\alpha}{2} \mathbf{V} = \mathbf{u}_0, & \mathbf{K}_g &= \frac{\partial \tilde{L}}{\partial \mathbf{j}_g} = \frac{\partial L}{\partial \mathbf{j}_g} + \frac{\partial L}{\partial \dot{\tau}} \frac{\nabla \tau}{\rho_g}, \\ R_l &= \frac{\partial \tilde{L}}{\partial \rho_l} = \frac{\partial L}{\partial \rho_l} + \frac{\delta L}{\delta \tau} \frac{\partial \tau}{\partial \rho_l} = \frac{\partial L}{\partial \rho_l} - \frac{1}{\rho \rho_g} \frac{\delta L}{\delta \tau}, \\ R_g &= \frac{\partial \tilde{L}}{\partial \rho_g} = \frac{\partial L}{\partial \rho_g} + \frac{\delta L}{\delta \tau} \frac{\partial \tau}{\partial \rho_g} - \frac{1}{\rho_g} \frac{\partial L}{\partial \dot{\tau}} \nabla \tau \cdot \mathbf{V} = \frac{\partial L}{\partial \rho_g} + \frac{\delta L}{\delta \tau} \frac{\partial \tau}{\partial \rho_g} - \frac{1}{\rho_g} \frac{\partial L}{\partial \dot{\tau}} \nabla \tau \cdot \mathbf{V}. \end{aligned}$$

Substitution of above expressions into (8.15) gives

$$\begin{aligned} \left( \mathbf{K}_\kappa \otimes \mathbf{j}_\kappa - \nabla \rho_\kappa \otimes \frac{\partial \tilde{L}}{\partial \nabla \rho_\kappa} \right) &= \sum_{\kappa=l,g} \frac{\partial L}{\partial \mathbf{j}_\kappa} \otimes \mathbf{j}_\kappa = \left( (1 - \alpha) \mathbf{U} - \frac{\alpha}{2} \mathbf{V} \right) \otimes \rho_l \mathbf{U} - \frac{\rho_l \tau}{2} (\mathbf{U} - \mathbf{V}) \otimes \rho_g \mathbf{V} \\ &= \rho_l \left( 1 - \frac{3\alpha}{2} \right) \mathbf{U} \otimes \mathbf{U} + \frac{\rho_l \alpha}{2} (\mathbf{U} - \mathbf{V}) \otimes (\mathbf{U} - \mathbf{V}). \end{aligned}$$

We see that the dyad part of flux (8.15) and (8.16) coincide. We can write the following expression for a spherical part of the flux (8.15)

$$\begin{aligned} \tilde{L} - \sum_{\kappa=l,g} (\rho_\kappa R_\kappa + \mathbf{K}_\kappa \cdot \mathbf{j}_\kappa) &= L - \rho_l \left( \frac{\partial L}{\partial \rho_l} + \frac{\delta L}{\delta \tau} \frac{\partial \tau}{\partial \rho_l} \right) - \rho_g \left( \frac{\partial L}{\partial \rho_g} + \frac{\delta L}{\delta \tau} \frac{\partial \tau}{\partial \rho_g} - \frac{1}{\rho_g} \frac{\partial L}{\partial \dot{\tau}} \nabla \tau \cdot \mathbf{V} \right) - \frac{\partial L}{\partial \mathbf{j}_l} \cdot \mathbf{j}_l - \frac{\partial L}{\partial \mathbf{j}_g} \cdot \mathbf{j}_g - \frac{\partial L}{\partial \dot{\tau}} \nabla \tau \cdot \mathbf{V} \\ &= L - \rho_l \frac{\partial L}{\partial \rho_l} - \rho_g \frac{\partial L}{\partial \rho_g} - \frac{\partial L}{\partial \mathbf{j}_l} \cdot \mathbf{j}_l - \frac{\partial L}{\partial \mathbf{j}_g} \cdot \mathbf{j}_g + \frac{1}{\rho_g} \frac{\delta L}{\delta \tau}. \end{aligned}$$



It has been shown that

$$\frac{1}{\rho_g} \frac{\delta L}{\delta \tau} = p - P_\infty + \rho \left( \frac{\partial L}{\partial \rho_l} + \frac{\mathbf{u}_0^2}{2} \right).$$

The spherical parts of (8.15), (8.16) coincide if

$$L - \rho_l \frac{\partial L}{\partial \rho_l} - \rho_g \frac{\partial L}{\partial \rho_g} - \frac{\partial L}{\partial \mathbf{j}_l} \cdot \mathbf{j}_l - \frac{\partial L}{\partial \mathbf{j}_g} \cdot \mathbf{j}_g + \rho \left( \frac{\partial L}{\partial \rho_l} + \frac{\mathbf{u}_0^2}{2} \right) = \frac{3}{8} \rho \alpha^2 \mathbf{V}^2. \quad (8.17)$$

Since the Lagrangian  $L(\rho_l, \rho_g, \mathbf{j}_l, \mathbf{j}_g, \tau, \dot{\tau})$  is a quadratic function of the momenta  $\mathbf{j}_k$  we obtain the equality

$$L - \frac{\partial L}{\partial \mathbf{j}_l} \cdot \mathbf{j}_l - \frac{\partial L}{\partial \mathbf{j}_g} \cdot \mathbf{j}_g = - \left( \left( 1 - \frac{3\tau\rho_g}{2} \right) \frac{\mathbf{j}_l^2}{2\rho_l} + \frac{\rho_l\tau\rho_g}{4} \left( \frac{\mathbf{j}_l}{\rho_l} - \frac{\mathbf{j}_g}{\rho_g} \right)^2 \right) + \frac{\rho\rho_g}{8\pi b} (\dot{\tau})^2 - \rho_g \varepsilon(\tau). \quad (8.18)$$

Remaining terms in (8.17) are evaluated as

$$\begin{aligned} & -\rho_l \frac{\partial L}{\partial \rho_l} - \rho_g \frac{\partial L}{\partial \rho_g} + \rho \left( \frac{\partial L}{\partial \rho_l} + \frac{\mathbf{u}_0^2}{2} \right) \\ &= \rho \alpha \frac{\partial L}{\partial \rho_l} - \rho_g \frac{\partial L}{\partial \rho_g} + \rho \frac{\mathbf{u}_0^2}{2} \\ &= \rho \alpha \left( - \left( 1 - \frac{3\alpha}{2} \right) \frac{\mathbf{U}^2}{2} + \frac{\alpha}{4} \left( \frac{\mathbf{j}_l}{\rho_l} - \frac{\mathbf{j}_g}{\rho_g} \right)^2 + \frac{\rho_l \alpha}{2} \left( \frac{\mathbf{j}_l}{\rho_l} - \frac{\mathbf{j}_g}{\rho_g} \right) \cdot \left( - \frac{\mathbf{j}_l}{\rho_l^2} \right) \right) \\ & \quad - \rho_g \left( - \frac{3\tau}{2} \frac{\mathbf{j}_l^2}{2\rho_l} + \frac{\rho_l \tau}{4} \left( \frac{\mathbf{j}_l}{\rho_l} - \frac{\mathbf{j}_g}{\rho_g} \right)^2 + \frac{\rho_l \rho_g \tau}{2} \left( \frac{\mathbf{j}_l}{\rho_l} - \frac{\mathbf{j}_g}{\rho_g} \right) \cdot \frac{\mathbf{j}_g}{\rho_g^2} + \frac{\rho}{8\pi b} (\dot{\tau})^2 - \varepsilon(\tau) \right) + \rho \frac{\mathbf{u}_0^2}{2}. \end{aligned} \quad (8.19)$$

Summing Eqs. (8.18) and (8.19) we derive equality (8.17). So, the momentum equations of the system (6.2) follow from the Hamilton principle.

It should be noted that the variational principle obtained can be also used for the description of rotational motions ( $\text{rot } \mathbf{K}_\kappa \neq 0$ ).

## 9. Linear waves

Consider small perturbations of the uniform state:

$$\mathbf{u}_0 = \delta \tilde{\mathbf{u}}_0, \quad \mathbf{V} = \delta \tilde{\mathbf{V}}, \quad p = p_0 + \tilde{p} = P_\infty + \tilde{p}, \quad n = n_0 + \delta \tilde{n}, \quad b = b_0 + \delta \tilde{b},$$

$\delta$  is a small parameter,  $b_0 = \text{const}$ ,  $n_0 = \text{const}$ ,  $p_0 = p_g(\tau_0)$ . The linearized system (6.2), (6.4) is

$$\begin{aligned} \tilde{n}_t + n_0 \text{div } \tilde{\mathbf{V}} &= 0, \quad 3\alpha_0 \tilde{b}_t = b_0 \text{div} \left( \tilde{\mathbf{u}}_0 + \frac{3\alpha_0}{2} \tilde{\mathbf{V}} \right), \quad \frac{\rho}{3} \left( 1 - \frac{3\alpha_0}{2} \right) \tilde{\mathbf{V}}_t + \nabla \tilde{p} = 0, \\ \rho \tilde{\mathbf{u}}_{0t} + \nabla \tilde{p} &= 0, \quad \text{rot } \tilde{\mathbf{u}}_0 = 0, \quad \tilde{b}_{tt} + \omega_r^2 \tilde{b} = - \frac{\tilde{p}}{\rho b_0}. \end{aligned} \quad (9.1)$$

Here

$$\omega_r^2 = - \frac{3\tau_0}{\rho b_0^2} \frac{dp_g}{d\tau}(\tau_0)$$

is a resonance bubble's frequency,  $\alpha_0 = n_0 \tau_0$ ,  $\tau_0 = 4/3b_0^3$ . The Fourier transform of (9.1) is:

$$\begin{aligned} \tilde{b}_t &= - \frac{b_0}{3\alpha_0} \left( \mathbf{i} \mathbf{k} \cdot \mathbf{u}_0 + \frac{3i\alpha_0}{2} \mathbf{k} \cdot \mathbf{V} \right), \quad \mathbf{V}_t = \frac{6i\mathbf{k}}{\rho(2-3\alpha_0)} p, \quad \mathbf{u}_{0t} = i\mathbf{k} \frac{p}{\rho}, \quad \mathbf{k} \wedge \mathbf{u}_0 = 0, \\ \tilde{b}_{tt} &= -\omega_r^2 \tilde{b} - \frac{p}{\rho b_0}, \quad n_t = in_0(\mathbf{k} \cdot \mathbf{V}). \end{aligned} \quad (9.2)$$

Here  $\mathbf{k}$  is the wave vector. The general solution of (9.2) is a linear combination of exponents  $e^{-i\omega t}$  with  $\omega$  being a root of the dispersion relation

$$\omega^2 = \omega_r^2 \frac{2b_0^2(1+3\alpha_0)|\mathbf{k}|^2}{3\alpha_0(2-3\alpha_0) + 2b_0^2(1+3\alpha_0)|\mathbf{k}|^2} \quad (9.3)$$

or  $\omega = 0$ . For  $0 < \alpha_0 < 2/3$  the dispersion relation (9.3) shows that  $\omega^2 < \omega_r^2$  for any finite  $|\mathbf{k}|$  and that  $\omega^2 \rightarrow \omega_r^2$  in a short wave limit ( $|\mathbf{k}| \rightarrow \infty$ ). The equilibrium sound velocity corresponding to the propagation of long waves ( $|\mathbf{k}| \rightarrow 0$ ) is:

$$C_e^2 = \lim_{|\mathbf{k}| \rightarrow 0} \frac{\omega^2}{|\mathbf{k}|^2} = b_0^2 \omega_r^2 \frac{2(1+3\alpha_0)}{3\alpha_0(2-3\alpha_0)}. \quad (9.4)$$

The phase velocity  $C = \omega/|\mathbf{k}|$  satisfies the inequality  $C^2(|\mathbf{k}|) < C_e^2$  for  $|\mathbf{k}| > 0$  and monotonically decreases to zero as  $|\mathbf{k}| \rightarrow \infty$ . For  $\alpha_0 > 2/3$  the wave modes become unstable for sufficiently small wave numbers.

It follows from (9.4) that  $C_e^2 \rightarrow +\infty$  as  $\alpha_0 \rightarrow 0$  (the limit of pure fluid) or  $\alpha_0 \rightarrow 2/3$  (the limit of the dense packing).

It is interesting to compare formula (9.4) with Wood's and Crespo's formulae:

$$C_{ew}^2 = \frac{\gamma p_0}{\rho \alpha_0(1-\alpha_0)}, \quad C_{ec}^2 = \frac{\gamma p_0(1+2\alpha_0)}{\rho \alpha_0(1-\alpha_0)}.$$

Here  $\gamma$  is the polytropic exponent. These formulae can be found, for example, in Miksis and Ting [18]. Formula (9.4) can be also rewritten as

$$C_e^2 = \frac{2(1+3\alpha_0)\gamma p_0}{\rho \alpha_0(2-3\alpha_0)}.$$

Expanding in small  $\alpha$  we have

$$C_{ew}^2 \approx \frac{\gamma p_0}{\rho \alpha_0}(1+\alpha_0), \quad C_{ec}^2 \approx \frac{\gamma p_0}{\rho \alpha_0}(1+3\alpha_0), \quad C_e^2 \approx \frac{\gamma p_0}{\rho \alpha_0} \left(1 + \frac{9}{2}\alpha_0\right).$$

We see that (9.4) gives a greater value of the sound velocity. Note that Wang and Smereka [29] have recently obtained effective hydrodynamical equations for bubbly liquids which give Crespo's formula in the long wave limit.

## 10. Motion of bubbles in a flow with uniform velocity

System (6.2), (6.4) describes the motion of the bubbles in the case when liquid velocity vanishes at infinity. To obtain the equations governing the motion of bubbles in a liquid which flows with a given constant velocity at infinity we apply Galilean transformation

$$\mathbf{x}' = \mathbf{x} + \mathbf{U}_\infty t, \quad \mathbf{U}' = \mathbf{U} + \mathbf{U}_\infty, \quad \mathbf{V}' = \mathbf{V} + \mathbf{U}_\infty \quad (10.1)$$

to Eqs. (6.2), (6.4). Here  $\mathbf{U}_\infty$  is the fluid velocity at infinity. The conservation law

$$F_t + \operatorname{div} \mathbf{G} = 0$$

transforms into

$$F_t + (\mathbf{U}_\infty \cdot \nabla) F + \operatorname{div} \mathbf{G} = F_t + \operatorname{div}(\mathbf{G} + F\mathbf{U}_\infty) = 0,$$

where  $F$  and  $\mathbf{G}$  have to be expressed through the variables with primes.

Transformed system governing bubbly flow is as follows (primes are omitted):

$$\begin{aligned} n_t + \operatorname{div}(n\mathbf{V}) &= 0, & (\rho(1-\alpha))_t + \operatorname{div}(\rho(1-\alpha)\mathbf{U}) &= 0, \\ \rho \left( \left( 1 - \frac{3\alpha}{2} \right) (1-\alpha)(\mathbf{U} - \mathbf{U}_\infty) \right)_t + \operatorname{div} \left( \rho \left( 1 - \frac{3\alpha}{2} \right) (1-\alpha)(\mathbf{U} - \mathbf{U}_\infty) \otimes \mathbf{U} + \frac{\rho\alpha(1-\alpha)}{2} (\mathbf{U} - \mathbf{V}) \otimes (\mathbf{U} - \mathbf{V}) \right) \\ &+ \nabla \left( p + \frac{3}{8} \rho \alpha^2 (\mathbf{V} - \mathbf{U}_\infty)^2 \right) &= 0, \\ \left( \alpha E + \frac{\rho}{2} (1-\alpha) \left( 1 - \frac{3\alpha}{2} \right) (\mathbf{U} - \mathbf{U}_\infty)^2 \right)_t \end{aligned}$$

$$\begin{aligned}
& + \operatorname{div} \left( \left( \alpha E + \frac{\rho}{2}(1-\alpha) \left( 1 - \frac{3\alpha}{2} \right) (\mathbf{U} - \mathbf{U}_\infty)^2 \right) \mathbf{V} + \left( p + \frac{3}{8} \rho \alpha^2 (\mathbf{V} - \mathbf{U}_\infty)^2 \right) (\mathbf{V} - \mathbf{U}_\infty) \right. \\
& \quad \left. + (1-\alpha) \left( p - P_\infty + \frac{1}{2} \rho \left( (1-\alpha) \mathbf{U} - \frac{\alpha}{2} \mathbf{V} - \left( 1 - \frac{3\alpha}{2} \right) \mathbf{U}_\infty \right)^2 \right) (\mathbf{U} - \mathbf{V}) \right) = 0.
\end{aligned} \quad (10.2)$$

Taking into account that (10.1) yields the following transformation law for  $\mathbf{u}_0$

$$\mathbf{u}'_0 = \mathbf{u}_0 + \left( 1 - \frac{3\alpha}{2} \right) \mathbf{U}_\infty = (1-\alpha) \mathbf{U} - \frac{\alpha}{2} \mathbf{V} + \left( 1 - \frac{3\alpha}{2} \right) \mathbf{U}_\infty$$

we rewrite Eqs. (6.4) in a new coordinate system (primes are omitted):

$$\begin{aligned}
& \rho \left( (1-\alpha) \mathbf{U} - \frac{\alpha}{2} \mathbf{V} + \frac{3\alpha}{2} \mathbf{U}_\infty \right)_t + \nabla \left( p + \frac{\rho}{2} \left( (1-\alpha) \mathbf{U} - \frac{\alpha}{2} \mathbf{V} + \frac{3\alpha}{2} \mathbf{U}_\infty \right)^2 \right) = 0, \\
& \operatorname{rot} \left( (1-\alpha) \mathbf{U} - \frac{\alpha}{2} \mathbf{V} + \frac{3\alpha}{2} \mathbf{U}_\infty \right) \Big|_{t=0} = 0.
\end{aligned}$$

It follows from these equations that there exists a potential  $\Phi$  such that

$$(1-\alpha) \mathbf{U} - \frac{\alpha}{2} \mathbf{V} + \frac{3\alpha}{2} \mathbf{U}_\infty = \nabla \Phi, \quad -\rho \left( \Phi_t + \frac{1}{2} (\nabla \Phi)^2 \right) = p - P_\infty - \frac{\rho \mathbf{U}_\infty^2}{2}. \quad (10.3)$$

The system (10.2)–(10.3) is Galilean invariant ( $\mathbf{U}_\infty$  transforms also). A generalized Rayleigh–Lamb’s equation (7.4) takes the form

$$b\dot{s} + \frac{3}{2}s^2 + \frac{p - p_g}{\rho} - \frac{(1-\alpha)^2}{4} (\mathbf{U} - \mathbf{V})^2 + \frac{3(1-\alpha)^2}{4} (\mathbf{U} - \mathbf{U}_\infty)^2 + \frac{3\alpha^2}{8} (\mathbf{V} - \mathbf{U}_\infty)^2 = 0.$$

## 11. Approximate hydrodynamical models

In the derivation of kinetic and hydrodynamical equations the series expansion of the flow potential in powers of small parameter  $\beta = \bar{b}/d = O(\alpha^{1/3})$  was used. The equations were derived with the error of order  $O(\beta^l)$ ,  $l > 3$ . To compare our hydrodynamical model with others, it is useful to expand (6.2), (6.3) in powers of  $\beta$ . Introducing dimensionless variables  $\mathbf{x}'$ ,  $b'$ ,  $t'$ ,  $\Phi'$ ,  $\mathbf{V}'$ ,  $s'$  similarly to (3.18), and  $\mathbf{U}'$ ,  $p'$ ,  $p'_g$ ,  $n'$ ,  $\alpha'$ , as follows:

$$\mathbf{U} = \beta^2 \left( \frac{P_\infty}{\rho} \right)^{1/2} \mathbf{U}', \quad p - P_\infty = \beta P_\infty p', \quad p_g = P_\infty p'_g, \quad n = \frac{1}{d^3} n', \quad \alpha = \beta^3 \alpha',$$

we rewrite Eqs. (6.2), (6.3) in dimensionless form

$$\begin{aligned}
& (1 - \beta^3 \alpha') \mathbf{U}' - \frac{\beta \alpha'}{2} \mathbf{V}' = \nabla' \Phi', \quad p' + \Phi'_{t'} + \frac{\beta^3}{2} (\nabla' \Phi')^2 = 0, \\
& n'_{t'} + \beta \operatorname{div}' (n' \mathbf{V}') = 0, \quad \alpha'_{t'} - \operatorname{div}' ((1 - \beta^3 \alpha') \mathbf{U}') = 0, \\
& ((\mathbf{V}' - 3\beta^2 (1 - \beta^3 \alpha') \mathbf{U}')_{t'} + \beta (\mathbf{V}' \cdot \nabla') (\mathbf{V}' - 3\beta^2 (1 - \beta^3 \alpha') \mathbf{U}')) + \frac{3s'}{b'} (1 - \beta^3 \alpha') (\mathbf{V}' - \beta^2 \mathbf{U}') \\
& \quad + 3\beta^3 (\mathbf{V}' \cdot \nabla') \left( (1 - \beta^3 \alpha') \mathbf{U}' - \frac{\beta \alpha'}{2} \mathbf{V}' \right) + \frac{3\beta^4}{8} \nabla' (\alpha'^2 \mathbf{V}'^2) = 0, \\
& b' s' + \frac{3}{2} s'^2 + 1 - p'_g + \beta p' - \frac{(1 - \beta^3 \alpha')^2}{4} (\mathbf{V}' - \beta^2 \mathbf{U}')^2 + \frac{3\beta^4 (1 - \beta^3 \alpha')^2}{4} \mathbf{U}'^2 + \frac{3\alpha'^2}{8} \beta^6 \mathbf{V}'^2 = 0, \\
& s' = b'_{t'} + \beta (\mathbf{V}' \cdot \nabla') b', \quad s'_{t'} + \beta (\mathbf{V}' \cdot \nabla') s' = 0.
\end{aligned} \quad (11.1)$$

We will distinguish “fast” and “slow” bubble motions. A “fast” motion is characterized by the asymptotics  $\mathbf{V} = O(1)$  or  $\mathbf{V} = O(\beta)$ . In a “slow” motion  $\mathbf{V} = O(\beta^2)$  (it is sufficient to fulfill these asymptotics at  $t = 0$ ). In the following we consider the “fast” motion  $\mathbf{V} = O(1)$  as a main flow, showing the changes in equations which appear for other asymptotics.

Equations of  $O(\beta^l)$  order of approximation are obtained by dropping of some  $O(\beta^{l+1})$  terms in (7.4). To a lower order of accuracy  $O(1)$  we derive the system

$$\begin{aligned}
& n_t = 0, \quad (\tau \mathbf{V})_t = 0, \quad \mathbf{U} = \nabla \Phi, \quad p - P_\infty = -\rho \Phi_t, \quad \Delta \Phi = \alpha_t, \\
& b\dot{s} + \frac{3}{2}s^2 + \frac{P_\infty - p_g}{\rho} - \frac{\rho}{4} \mathbf{V}^2 = 0, \quad s = b_t, \quad \dot{s} = s_t
\end{aligned} \quad (11.2)$$

that describes the motions of noninteracting bubbles (we have returned to dimensional variables in (11.2)). Integrating the first two equations of (11.2) we find

$$n = n_0(\mathbf{x}), \quad \mathbf{V} = \frac{\tau_0(\mathbf{x})}{\tau} \mathbf{V}_0(\mathbf{x}).$$

Here  $\tau_0(\mathbf{x})$ ,  $\mathbf{V}_0(\mathbf{x})$ ,  $n_0(\mathbf{x})$ , are the initial values of the volume, velocity and bubble number density. The bubble radius is obtained by integrating Rayleigh–Lamb’s equation. The pressure in liquid induced by the bubble motions can be found from (11.2) after integration of Poisson’s equation for  $\Phi$ . No influence of the induced pressure on the bubble motion is observed.

The  $O(\beta)$  approximation of (11.1) written in dimensional variables

$$\begin{aligned} n_t + \operatorname{div}(n\mathbf{V}) &= 0, & (\tau\mathbf{V})_t + (\mathbf{V} \cdot \nabla)(\tau\mathbf{V}) &= 0, \\ \mathbf{U} - \frac{\alpha}{2}\mathbf{V} &= \nabla\Phi, & p - P_\infty &= -\rho\Phi_t, & \operatorname{div}\mathbf{U} &= \alpha_t, \\ b\dot{s} + \frac{3}{2}s^2 + \frac{p - p_g}{\rho} - \frac{1}{4}\mathbf{V}^2 &= 0, & s &= b_t + (\mathbf{V} \cdot \nabla)b, & \dot{s} &= s_t + (\mathbf{V} \cdot \nabla)s \end{aligned} \quad (11.3)$$

accounts for induced pressure effect on the bubble radius oscillations. For the case  $\mathbf{V} = O(\beta^l)$ ,  $l = 1, 2$ , we can simplify system (11.3) to

$$\begin{aligned} n_t &= 0, & (\tau\mathbf{V})_t &= 0, & \mathbf{U} &= \nabla\Phi, & p - P_\infty &= -\rho\Phi_t, & \Delta\Phi &= \alpha_t, & b\dot{s} + \frac{3}{2}s^2 + \frac{p - p_g}{\rho} &= 0, \\ s &= b_t, & \dot{s} &= s_t. \end{aligned} \quad (11.4)$$

We obtain from (11.4) the system

$$4\pi\rho n_0(b^2s)_t = -\Delta p, \quad \rho\left(bs_t + \frac{3}{2}s^2\right) - p_g = -p \quad (11.5)$$

coinciding in the limit of incompressible liquid with approximate equations derived by Caflisch et al. [4]. System (11.5) reduces to Foldy’s equations [6] under assumption that the bubble oscillations are small.

More detailed description of the induced pressure and added mass effects is attained if we use  $O(\beta^2)$  approximation of (11.1):

$$\begin{aligned} n_t + \operatorname{div}(n\mathbf{V}) &= 0, & \left(\frac{\tau}{2}(\mathbf{V} - \mathbf{U})\right)_t + (\mathbf{V} \cdot \nabla)\left(\frac{\tau}{2}(\mathbf{V} - \mathbf{U})\right) - \tau(\mathbf{U}_t + (\mathbf{V} \cdot \nabla)\mathbf{U}) &= 0, \\ \mathbf{U} - \frac{\alpha}{2}\mathbf{V} &= \nabla\Phi, & p &= P_\infty - \rho\Phi_t, & \operatorname{div}(\mathbf{U} + \alpha\mathbf{V}) &= \frac{3\alpha s}{b}, \\ b\dot{s} + \frac{3}{2}s^2 + \frac{p - p_g}{\rho} - \frac{1}{4}(\mathbf{V} - \mathbf{U})^2 &= 0. \end{aligned} \quad (11.6)$$

In the case of the “slow” bubble motion we can rewrite (11.6) in the form

$$\begin{aligned} n_t &= 0, & \left(\frac{\tau}{2}(\mathbf{V} - \mathbf{U})\right)_t - \tau\mathbf{U}_t &= 0, & \mathbf{U} &= \nabla\Phi, & \rho\Phi_t + \frac{\rho}{2}(\nabla\Phi)^2 + p &= P_\infty, \\ (1 - \alpha)_t + \operatorname{div}((1 - \alpha)\mathbf{U}) &= 0, & b\dot{s} + \frac{3}{2}s^2 + \frac{p - p_g}{\rho} &= 0. \end{aligned} \quad (11.7)$$

The last three equations can be rewritten as

$$\tilde{\rho}_t + \operatorname{div}(\tilde{\rho}\mathbf{U}) = 0, \quad \rho(\mathbf{U}_t + (\mathbf{U} \cdot \nabla)\mathbf{U}) + \nabla p = 0, \quad p = p_g - \rho\left(b\dot{s} + \frac{3}{2}s^2\right). \quad (11.8)$$

Here the mean density of the mixture  $\tilde{\rho}$  is defined by the formula  $\tilde{\rho} = (1 - \alpha)\rho$  because the gas mass is neglected. System (11.8) is very close to Iordansky–Kogarko–van Wijngaarden system [12,14,30] because of the estimate  $\tilde{\rho} - \rho = O(\beta^3)$ . The second equation of (11.7) coincides to  $O(\beta^2)$  order with the momentum equation of gas phase proposed by Biesheuvel and van Wijngaarden [2].

Next we consider system (10.2), (10.3) which is more suitable for comparison with the models describing motions of the bubbles in an external flow. For the case when  $\mathbf{U}$  does not vanish at infinity, we use the following scaling for  $\Phi$ ,  $\mathbf{U}$  and  $p$ :

$$\Phi = d\left(\frac{P_\infty}{\rho}\right)^{1/2}\Phi', \quad \mathbf{U} = \left(\frac{P_\infty}{\rho}\right)^{1/2}\mathbf{U}', \quad \mathbf{U}_\infty = \left(\frac{P_\infty}{\rho}\right)^{1/2}\mathbf{U}'_\infty, \quad p = P_\infty p'.$$

Other variables are scaled similarly to (11.2). Here we can also derive approximate systems expanding nondimensional Eqs. (10.2), (10.3) in powers of  $\beta$ . It follows from these equations that

$$\mathbf{U} - \mathbf{U}_\infty = O(\beta^2), \quad \nabla\Phi - \mathbf{U}_\infty = O(\beta^2), \quad \left(1 - \frac{3\alpha}{2}\right)\nabla\Phi = (1 - \alpha)\mathbf{U} - \frac{\alpha}{2}\mathbf{V} + O(\beta^5).$$

Neglecting some terms of order  $O(\beta^4)$  and higher we write the hydrodynamical system in the form

$$\begin{aligned} n_t + \operatorname{div}(n\mathbf{V}) &= 0, & (1 - \alpha)_t + \operatorname{div}((1 - \alpha)\mathbf{U}) &= 0, \\ p + \rho\Phi_t + \frac{\rho}{2}(\nabla\Phi)^2 &= P_\infty + \frac{\rho}{2}\mathbf{U}_\infty^2, & bs + \frac{3}{2}s^2 + \frac{p - p_g}{\rho} - \frac{(1 - \alpha)^2}{4}(\mathbf{V} - \mathbf{U})^2 &= 0, \\ \left(\frac{\partial}{\partial t} + (\mathbf{V} \cdot \nabla)\right) \left(\frac{\tau(1 - \alpha)}{2}(\mathbf{V} - \mathbf{U})\right) - \tau \left(\frac{\partial}{\partial t} + \frac{1}{2}(3\mathbf{U}_\infty - \mathbf{V}) \cdot \nabla\right) (\nabla\Phi - \mathbf{U}_\infty) &= 0. \end{aligned} \quad (11.9)$$

If the bubble velocity is such that  $\mathbf{V} - \mathbf{U}_\infty = O(\beta^l)$ ,  $l \geq 1$ , we obtain, after dropping some  $O(\beta^3)$  terms, Rayleigh–Lamb’s equation and the momentum equation of gas phase in the form

$$\begin{aligned} bs + \frac{3}{2}s^2 + \frac{p - p_g}{\rho} - \frac{1}{4}(\mathbf{V} - \mathbf{U})^2 &= 0, \\ \left(\frac{\partial}{\partial t} + (\mathbf{V} \cdot \nabla)\right) \left(\frac{\tau}{2}(\mathbf{V} - \mathbf{U})\right) - \tau \left(\frac{\partial}{\partial t} + (\nabla\Phi \cdot \nabla)\right) \nabla\Phi &= 0 \end{aligned}$$

proposed by Voinov and Petrov [28].

Eqs. (11.9) are also close to the averaged hydrodynamical equations proposed by Biesheuvel and van Wijngaarden [2]. Rayleigh–Lamb’s equation is the same as that of Biesheuvel and van Wijngaarden. The momentum equation of bubbles is in the form similar to the fifth equation of (11.9) if we neglect some  $O(\beta^3)$  terms. However, there is a difference in the assumptions which were used in the derivation of the discussed systems. Our hydrodynamical system describes motions in the bubble cloud in the case when velocity at infinity vanishes or tends to constant. If some flux terms in the momentum equations of liquid phase in the Biesheuvel–van Wijngaarden model were derived through calculation of fluctuation terms, system (10.2), (10.3) was obtained as an exact consequence of kinetic equations (3.14)–(3.17) with fluctuation terms vanishing identically (see (5.8), (5.10) and (6.1)) in a chosen class of solutions. It means that the comparison of model (10.2), (10.3) with Biesheuvel–van Wijngaarden’s equations is correct only for regular flows characterized by very small fluctuations of flow parameters. In this class we have a good agreement between the models. Notice that the kinetic system has another classes of solutions describing the motions with great fluctuations.

Summarizing, we can conclude that the hydrodynamical system derived describing particular solutions of kinetic equations is asymptotically close to known models of bubbly flow.

## 12. Conclusion

We have derived a kinetic model governing the evolution of gas–liquid mixtures in the case of an ideal fluid and compressible gas bubbles. The derivation is based on an approximate calculation of the kinetic energy (Hamiltonian of the system) which is correct to  $\beta^3 = O(\alpha)$  where  $\beta$  is the ratio of the mean bubble radius to the mean distance between bubbles,  $\alpha$  is a void fraction. The Vlasov approach has been used to reduce the kinetic description to an equation for the one-particle distribution function and a system for a self-consistent field describing the mean liquid flow. No special closure relations have been exploited except for general assumptions commonly used in the derivation of kinetic models. For the limit case of the rigid balls the kinetic equations by Russo and Smereka [22] and Herrero et al. [11] were discovered. The exact conservation laws of the mass, momentum and energy were obtained from the Vlasov system.

Hydrodynamical equations have been derived as a consequence of the kinetic equations in a special class of solutions. It has been shown that the hydrodynamical model can be also obtained as Euler–Lagrange equations for Hamilton’s action with a special Lagrangian. The equations commonly used for the description of bubbly fluids (models proposed by Caflisch et al. [4], Foldy [6], Voinov and Petrov [28], Iordansky [12], Kogarko [14], van Wijngaarden [30], Biesheuvel and van Wijngaarden [2]) have been discovered in the limit of small volume concentrations.

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## Appendix A

Here we find in explicit form the coefficients of quadratic form (2.11). When evaluating the integrals over the  $i$ th sphere in the sums in Eqs. (1.7) and (1.9), we need to consider only the terms  $\psi_{ji}$  and  $\varphi_{ji}$  since the contribution of the remaining terms is of order higher than  $\beta^3$ . It follows from (2.8)–(2.10) that

$$\psi_j|_{\Gamma_i} = \begin{cases} \frac{b_j^3}{2r_{ij}^2} \mathbf{n}_{ji} - \frac{3}{4} \left( \frac{b_j}{r_{ij}} \right)^3 B_{ij} (\mathbf{x} - \mathbf{x}_i) + O(\beta^4), & i \neq j, \\ -\frac{\mathbf{x} - \mathbf{x}_j}{2} + O(\beta^4), & i = j. \end{cases}$$

Substituting these expressions into (2.11) and dropping higher-order terms we obtain

$$A_{ji} = \frac{b_j^3}{2r_{ij}^2} \mathbf{n}_{ji} \otimes \iint_{\Gamma_i} \mathbf{n}_i d\Gamma - \frac{3}{4} \left( \frac{b_j}{r_{ij}} \right)^3 B_{ij} \iint_{\Gamma_i} (\mathbf{x} - \mathbf{x}_i) \otimes \mathbf{n}_i d\Gamma = -\pi \frac{b_j^3 b_i^3}{r_{ij}^3} B_{ij}, \quad i \neq j \quad \text{and} \quad A_{jj} = -\frac{\tau_j}{2} I,$$

since

$$\iint_{\Gamma_i} \mathbf{n}_i d\Gamma = 0, \quad \iint_{\Gamma_i} (\mathbf{x} - \mathbf{x}_i) \otimes \mathbf{n}_i d\Gamma = \tau_i I. \quad (\text{A.1})$$

Using (A.1) we evaluate coefficients

$$\mathbf{c}_{ij} = \frac{b_j^3}{2r_{ij}^2} \mathbf{n}_{ji} \iint_{\Gamma_i} d\Gamma - \frac{3}{4} \left( \frac{b_j}{r_{ij}} \right)^3 B_{ij} \iint_{\Gamma_i} \mathbf{n}_i d\Gamma = 2\pi \frac{b_i^2 b_j^3}{r_{ij}^2} \mathbf{n}_{ji}, \quad i \neq j, \quad \mathbf{c}_{jj} = -\frac{b_j}{2} \iint_{\Gamma_j} \mathbf{n}_j d\Gamma = 0.$$

In derivation of the expression for  $\mathbf{d}_{ij}$  and  $e_{ij}$  we use approximations with remainder  $O(\beta^4)$  for the trace of  $\varphi_j$  on  $\Gamma_i$  (see (2.4), (2.5), (2.7)):

$$\varphi_j|_{\Gamma_i} = -\frac{b_j^2}{r_{ij}} \sum_{n=0}^2 \frac{2n+1}{n+1} \left( \frac{b_i}{r_{ij}} \right)^n P_n(\cos \theta_{ji}), \quad j \neq i, \quad \varphi_j|_{\Gamma_j} = -b_j. \quad (\text{A.2})$$

To calculate integrals over  $\Gamma_i$  it is convenient to use the spherical coordinates  $r_i, \theta_{ji}, \lambda_{ji}$  with  $z$ -axis directed along  $\mathbf{n}_{ji}$ :

$$\mathbf{x} - \mathbf{x}_i = r_i (\sin \theta_{ji} \cos \lambda_{ji}, \sin \theta_{ji} \sin \lambda_{ji}, \cos \theta_{ji}) = r_i \mathbf{n}_i.$$

Taking into account that  $\varphi_j$  does not depend on  $\lambda_{ji}$ , we write

$$\mathbf{d}_{ij} = \iint_{\Gamma_i} \varphi_j \mathbf{n}_i d\Gamma = b_i^2 \int_0^\pi \varphi_j \sin \theta_{ji} \left( \int_0^{2\pi} \mathbf{n}_i d\lambda_{ji} \right) d\theta_{ji} = 2\pi b_i^2 \mathbf{n}_{ji} \int_0^\pi \varphi_j \cos \theta_{ji} \sin \theta_{ji} d\theta_{ji}.$$

Substituting the expression for  $\varphi_j$ , we observe that only one term in sum (A.2) corresponding to  $n = 1$  gives nonzero contribution to the integral because of the orthogonality of  $P_n(\cos \theta)$  to  $P_1(\cos \theta) = \cos \theta$  for  $n \neq 1$ . Then we find

$$\mathbf{d}_{ij} = -3\pi \frac{b_i^3 b_j^2}{r_{ij}^2} \mathbf{n}_{ji} \int_0^\pi \cos^2 \theta \sin \theta d\theta = -2\pi \frac{b_i^3 b_j^2}{r_{ij}^2} \mathbf{n}_{ji}, \quad i \neq j.$$

Obviously,  $\mathbf{d}_{jj} = 0$ . Similarly, we can take into account only one term in (A.1) corresponding to  $n = 0$  in calculation of the integral:

$$e_{ij} = \iint_{\Gamma_i} \varphi_j d\Gamma = 2\pi b_i^2 \int_0^\pi \varphi_j \sin \theta d\theta = -4\pi \frac{b_i^2 b_j^2}{r_{ij}}, \quad i \neq j,$$

because of the orthogonality of  $P_n(\cos \theta) = \cos \theta$  and  $P_0(\cos \theta) = 1$ . For  $i = j$  we get

$$e_{jj} = -4\pi b_j^3.$$

Using the formulae obtained we can write the expression (2.12) for the kinetic energy.

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